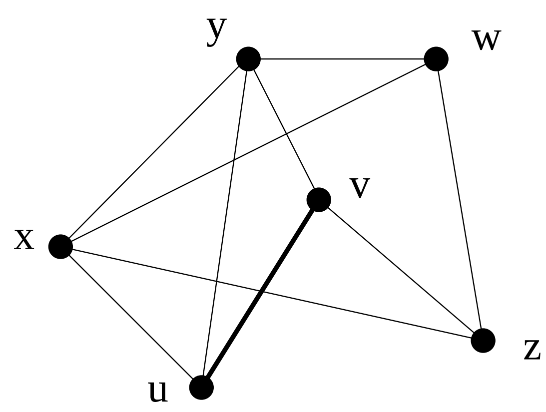
Geometric Graphs

Sathish Govindarajan
Indian Institute of Science, Bangalore

Workshop on Introduction to Graph and Geometric Algorithms

Geometric Graph



☆ V = set of geometric objects (point set in the plane)

☆ $E = \{(u, v)\}$ based on some geometric condition

Questions on Geometric Graphs

- Problems on graphs
  - Independent set, coloring, clique, etc.

- Combinatorial/Structural questions
  - Obtain Bounds
  - Characterization

- Computational questions
  - Efficient Algorithm
  - Approximation

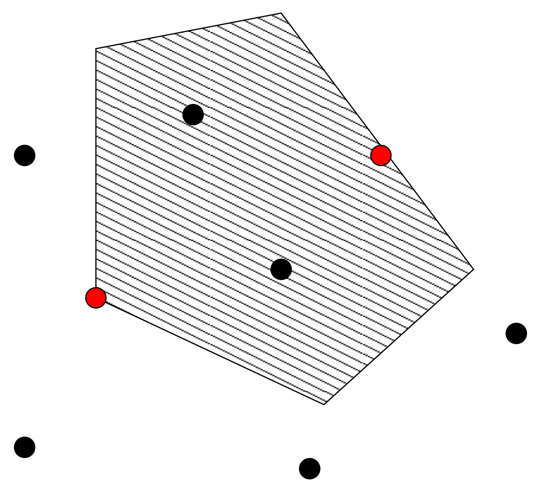
Geometric graphs

- V - set of geometric objects
- E - object i and j satisfy certain geometric condition

- Broad classes of geometric graphs (based on edge condition)
  - Proximity graphs
  - Intersection graphs
  - Distance based graphs

Proximity Graphs

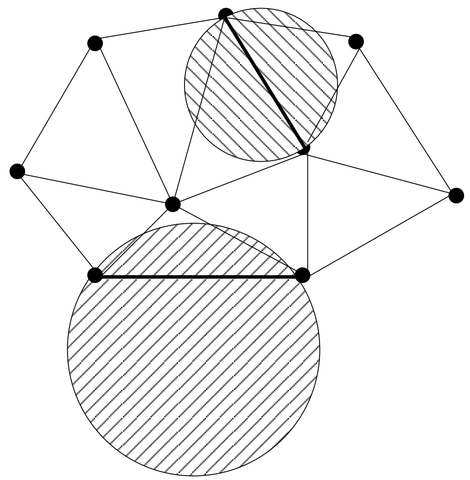
- ☆ P - point set in plane
- ☆ $R_{i,j}$ - proximity region defined by i and j



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $R_{i,j}$ is empty
- ☆ Examples - Delaunay, Gabriel, Relative Neighborhood Graph
- ☆ Applications - Graphics, wireless networks, GIS, computer vision, etc.

Delaunay Graph - Classic Example

☆ P - point set in plane



☆ V - point set P

☆ $(i, j) \in E$ if $\exists$ some empty circle thro' i and j

☆ Triangle (i, j, k) if circumcircle(i, j, k) is empty
(Equivalent condition)

☆ Applications - Graphics, mesh generation, computer vision, etc.

Questions on Delaunay Graph

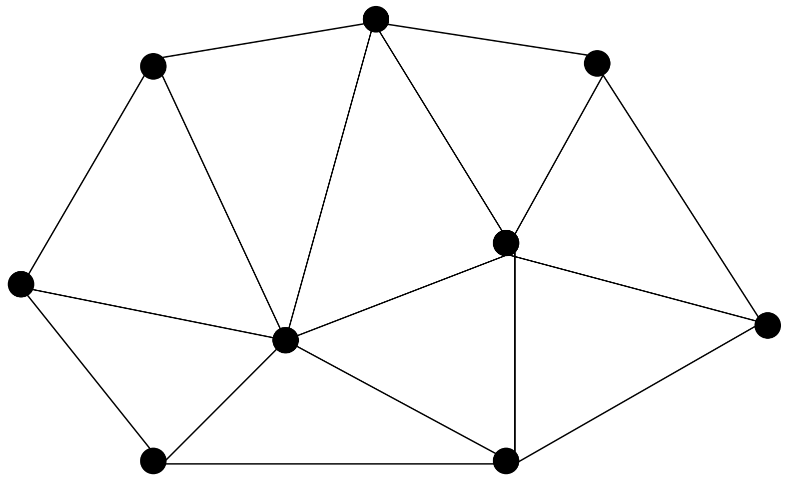
- Combinatorial - Bounds on
  - Maximum size of edge set?
  - Chromatic number?
  - Maximum independent set?

  (Over all possible point sets P)

- Computational
  - Efficient Algorithm

Delaunay Graph - Classic Example

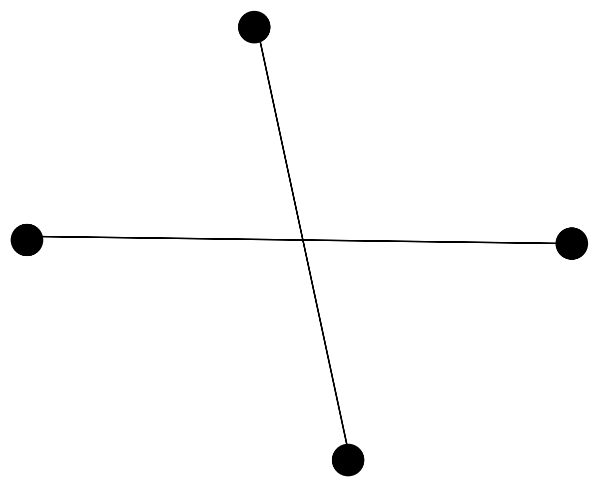
☆ P - point set in plane



☆ Observations: Planar?

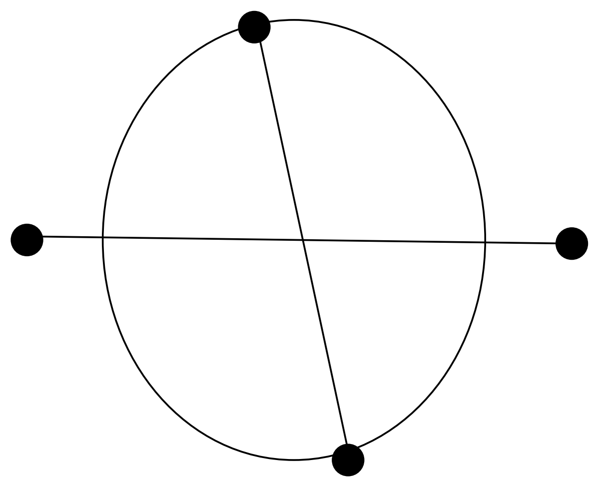
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



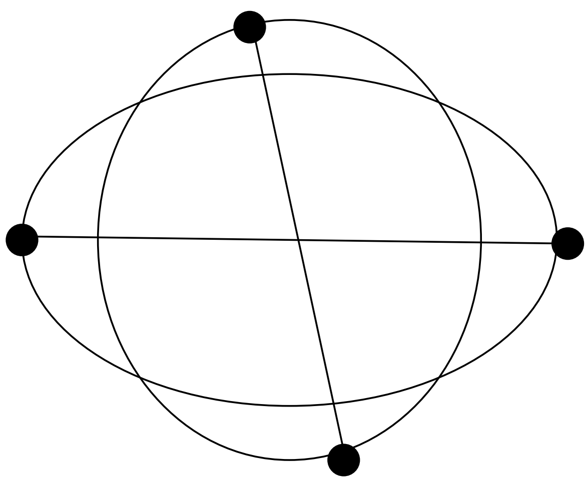
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



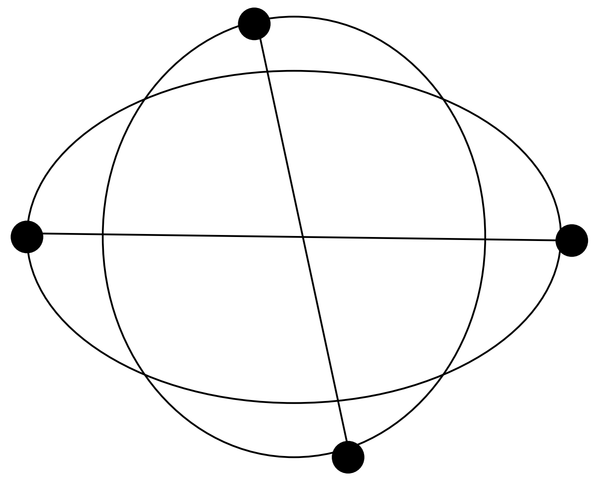
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



Delaunay Graph - Planar

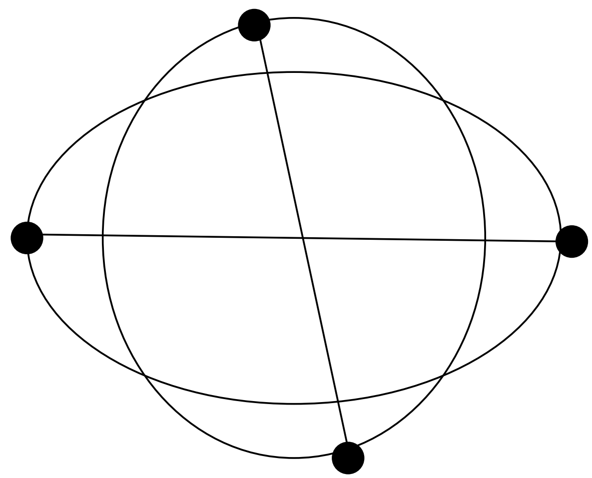
★ Let, if possible, 2 edges cross



★ Circles c'ant intersect like this (why?)

Delaunay Graph - Planar

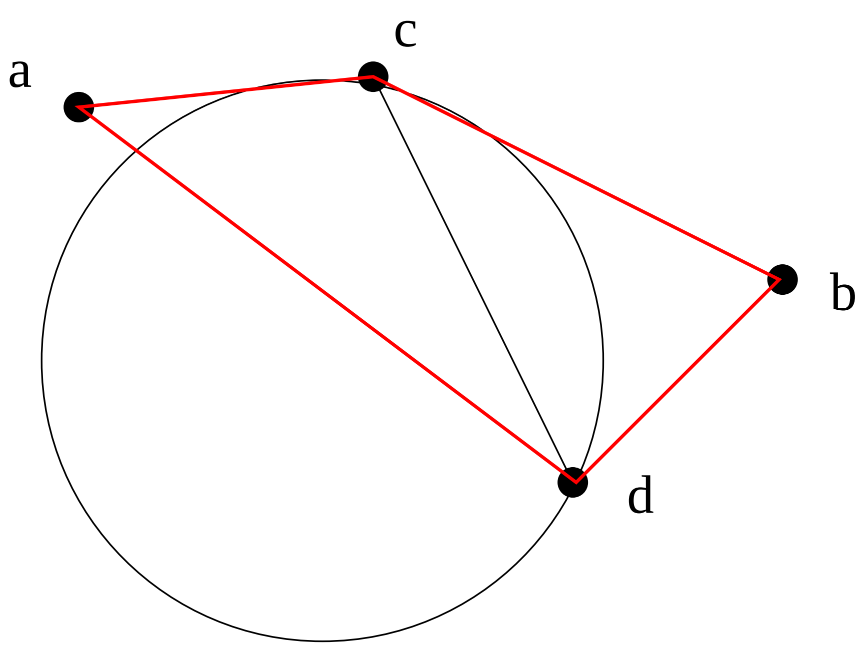
- Let, if possible, 2 edges cross



- Circles c'ant intersect like this (why?)
- One endpoint of an edge lies within the other circle
  - Contradiction

Delaunay Graph - Formal Proof

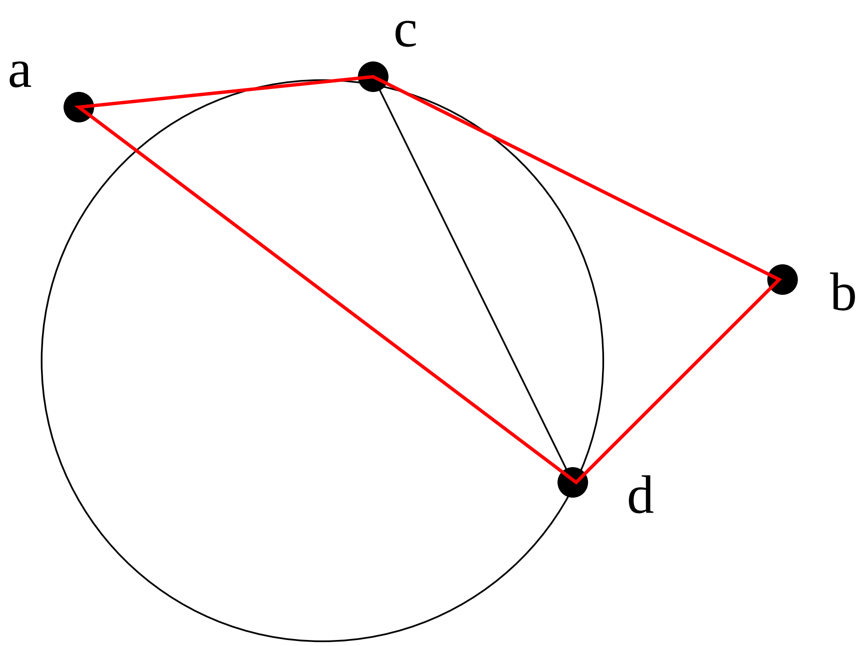
- ☆ Consider any circle passing through c and d
- ☆ Points a and b are outside the circle



- ☆ What about $\angle cad + \angle cbd$?

Delaunay Graph - Formal Proof

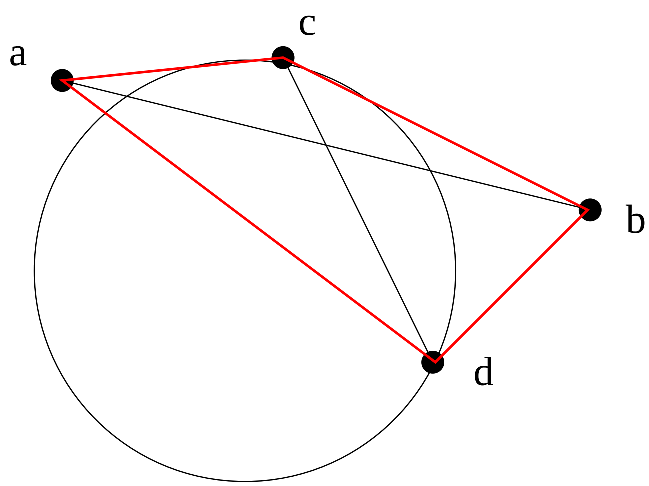
- ☆ Consider any circle passing through c and d
- ☆ Points a and b are outside the circle



- ☆ $\angle cad + \angle cbd < 180$

Delaunay Graph - Formal Proof

- Let, if possible, edges ab and cd cross
- Consider the quadrilateral $acdb$



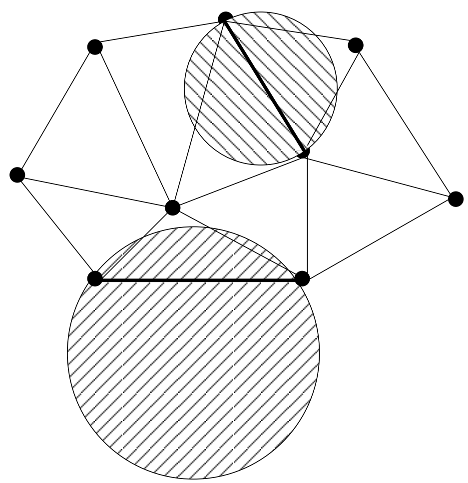
- cd is an edge $\implies \angle cad + \angle cbd < 180$
- ab is an edge $\implies \angle acb + \angle adb < 180$
- $\angle cad + \angle cbd + \angle acb + \angle adb < 360$
  - Contradiction

Questions on Delaunay Graph

- Given any n-point set P in the plane
  - Delaunay graph is planar
- Maximum size of edge set
  - $\leq 3n - 6$ (Euler's formula)
- Chromatic number
  - ≤ 4 (Four color theorem)
- Maximum independent set
  - $\geq n/4$ (Chromatic number)

Delaunay Graph

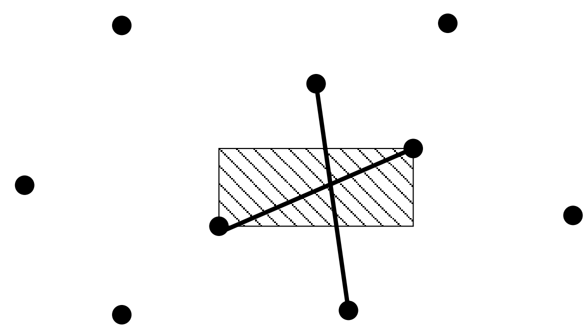
- ☆ P - point set in plane



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $\exists$ some empty **circle** thro' i and j

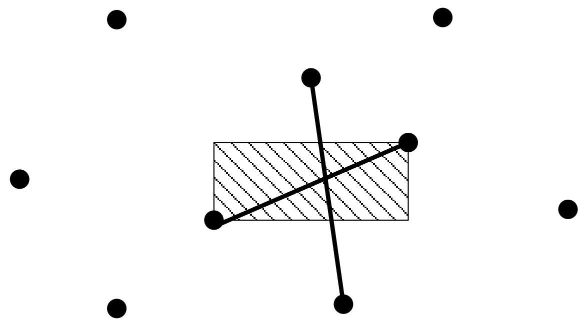
Delaunay Graph - Variants

- ⋆ Edges defined by other objects (instead of circles)
- ⋆ Objects - Square, Halfspace, Ellipse, Rectangle
- ⋆ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j



Delaunay Graph wrt Rectangles

⋆ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j

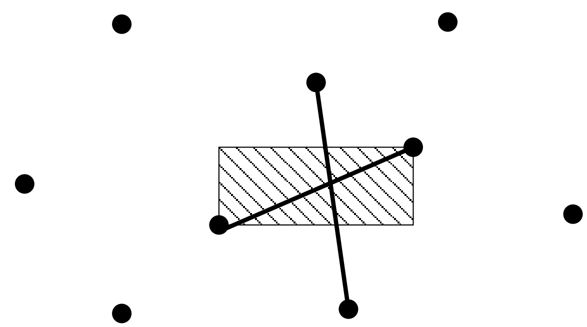


⋆ Bounds on the size of maximum independent set?

⋆ Application: Frequency assignment in wireless networks

Delaunay Graph wrt Rectangles

★ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j



★ Graph Properties

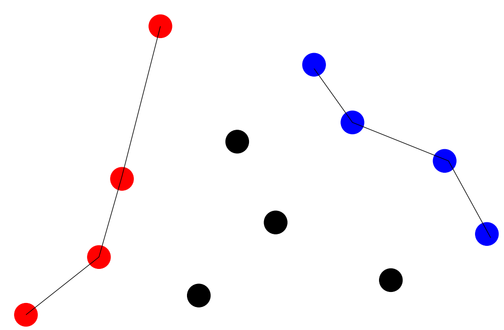
- Graph can have $\Omega(n^2)$ edges
- $K_n, n \geq 5$ is a forbidden subgraph

Bounds on Independent Set Size

Theorem: Any Delaunay graph (wrt rectangles) has an independent set of size atleast $\sqrt{n}/2$

Bounds on Independent Set Size

- Same slope sequence of points
  - +ve slope sequence (Red)
  - -ve slope sequence (Blue)



- Same slope sequence of size $2k$
  - Independent set of size k

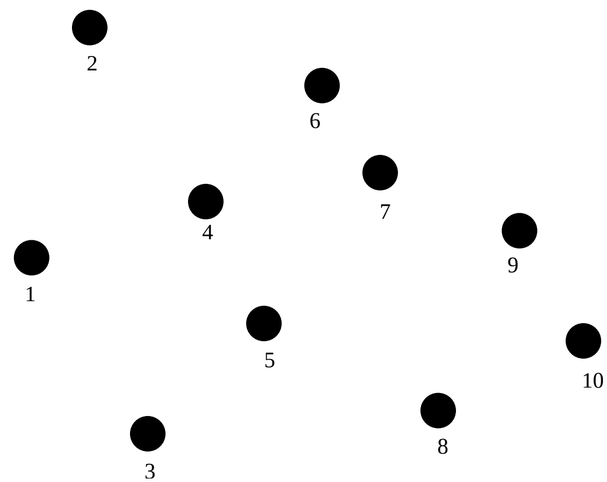
Bounds on Independent Set Size

Theorem: Let P be any set of $m^2 + 1$ points in the plane.
There exists a same slope sequence (+ve or -ve) of size $m + 1$.

- ⋆ Erdos and Szekeres proved it in 1935
- ⋆ Atleast six different proofs
  (Monotone subsequence survey by Michael Steele)

Bounds on Independent Set Size

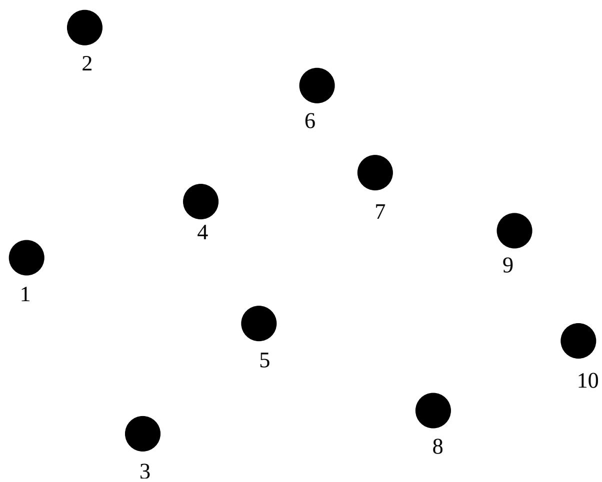
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1
- l_2 :
- l_3 :

Bounds on Independent Set Size

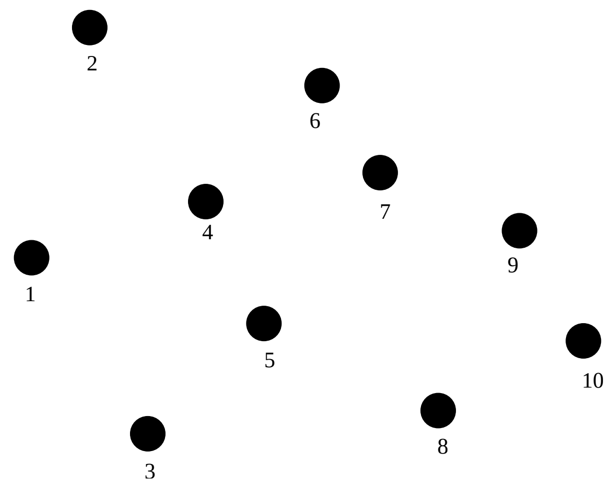
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1
- l_2 : 2
- l_3 :

Bounds on Independent Set Size

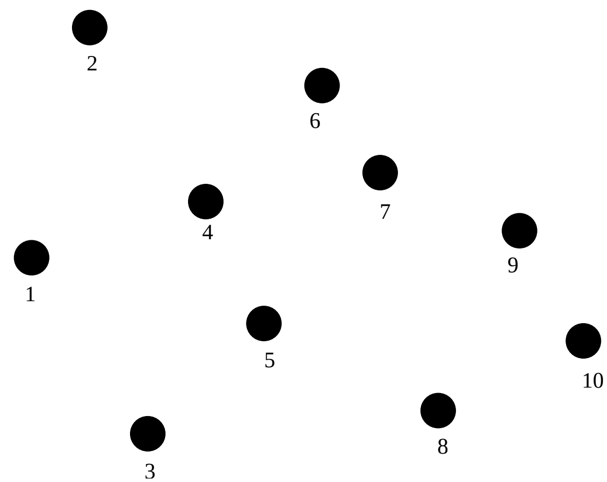
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1, 3
- l_2 : 2
- l_3 :

Bounds on Independent Set Size

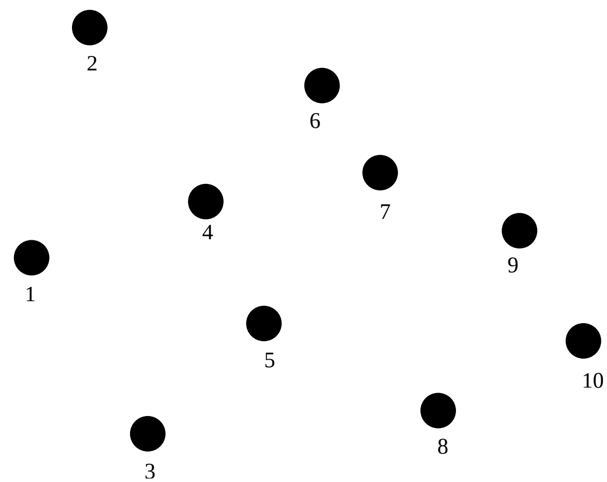
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1, 3
- l_2 : 2, 4
- l_3 :

Bounds on Independent Set Size

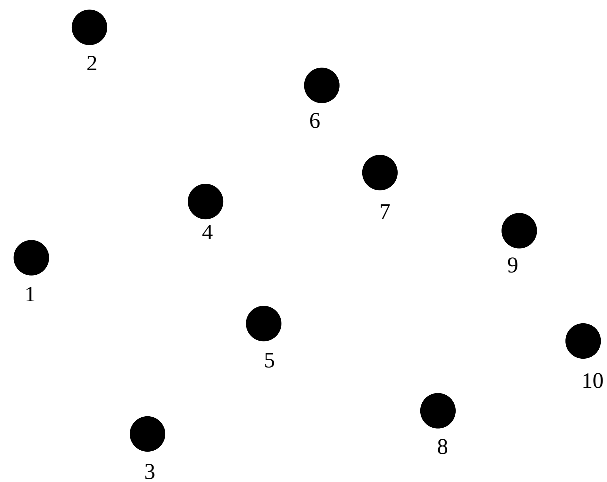
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1, 3
- l_2 : 2, 4, 5
- l_3 :

Bounds on Independent Set Size

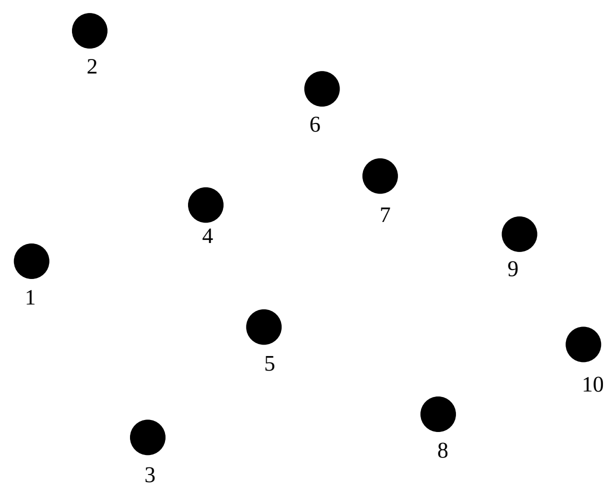
- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1, 3
- l_2 : 2, 4, 5
- l_3 : 6

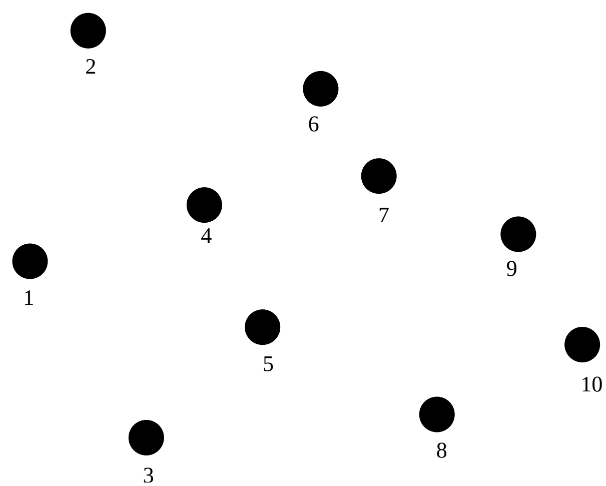
Bounds on Independent Set Size

- Partition the points into lists $l_1, l_2, \ldots, l_k$
  - For each point p, find the longest +ve slope sequence ending in p
  - Let c be the length of this sequence
  - Place p in list l_c



- l_1 : 1, 3
- l_2 : 2, 4, 5, 8
- l_3 : 6, 7, 9, 10

Bounds on Independent Set Size



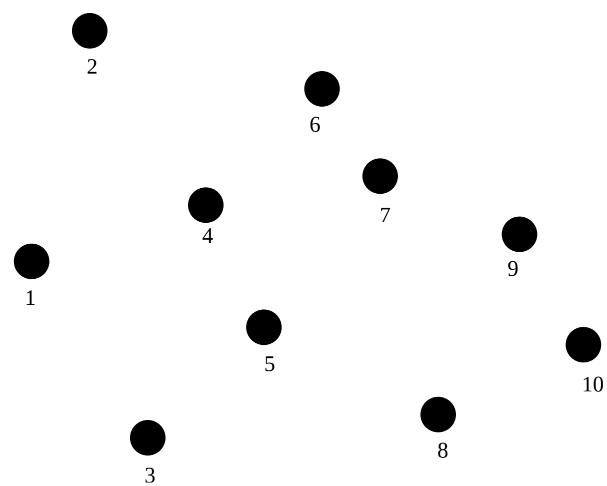
- ☆ l_1 : 1, 3
- ☆ l_2 : 2, 4, 5, 8
- ☆ l_3 : 6, 7, 9, 10

- ☆ Observation: Sequence of points in a list is -ve slope sequence
  - Proof?

Bounds on Independent Set Size

Theorem: Let P be any set of $m^2 + 1$ points in the plane. There exists a same slope sequence (+ve or -ve) of size $m + 1$.

- ☆ Partition the points into lists $l_1, l_2, \ldots, l_k$
- ☆ If $k \geq m + 1$, we are done.



- ☆ l_1 : 1, 3
- ☆ l_2 : 2, 4, 5, 8
- ☆ l_3 : 6, 7, 9, 10

Bounds on Independent Set Size

Theorem: Let P be any set of $m^2 + 1$ points in the plane. There exists a same slope sequence (+ve or -ve) of size $m + 1$.

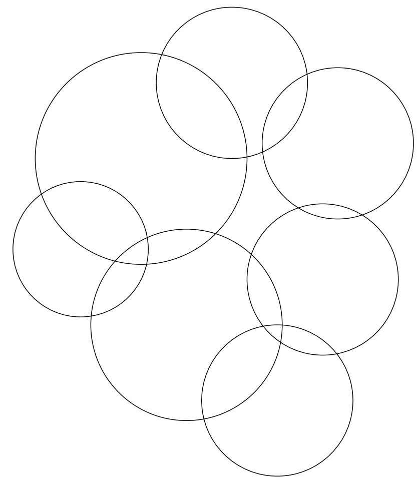
- ☆ Partition the points into lists $l_1, l_2, \ldots, l_k$
- ☆ If $k \geq m + 1$, we are done.
- ☆ Otherwise, one of the list has alteast $m + 1$ points
  - There are $\leq m$ lists and $m^2 + 1$ points
  - Apply Pigeon hole principle
- ☆ By Observation, there is a -ve slope sequence of size atleast $m+1$

Independent Set - Open Problem

- Size of maximum independent set - Lower bound
  - $\Omega(n^{0.5})$ (Slope sequence)
  - Improved to $\Omega(n^{0.618-\epsilon})$ (Ajwani et al, SPAA '07)
- Size of maximum independent set - Upper bound
  - $O(n/\log n)$ (Pach et al '08)
- Conjecture: Close to $O(n/\log n)$
- Open problem : Obtain better upper/lower bounds

Intersection Graphs

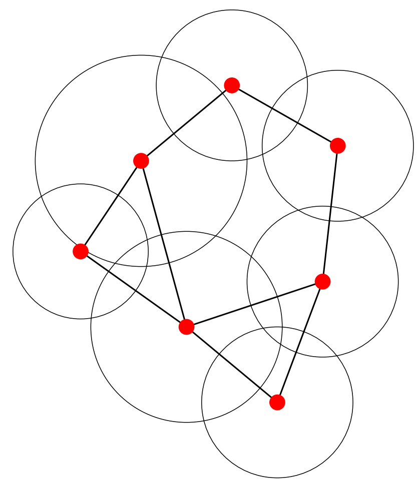
- ☆ S - set of geometric objects s_i(circles)



- ☆ V - set of object s_i
- ☆ $(s_i, s_j) \in E$ if objects s_i and s_j intersect

Intersection Graphs

- S - set of geometric objects s_i(circles)



- $(s_i, s_j) \in E$ if objects s_i and s_j intersect

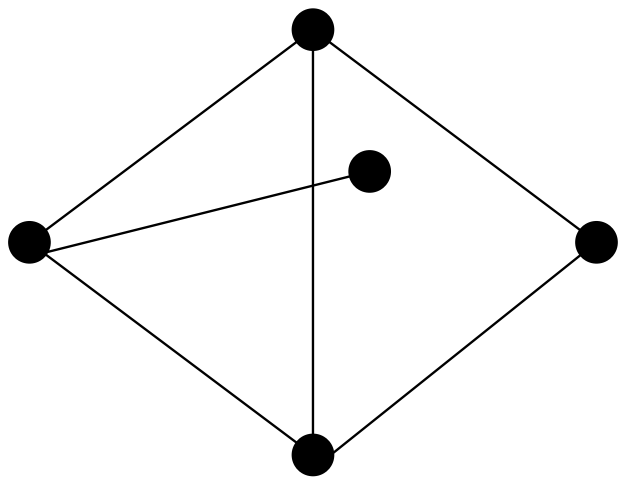
- Graph problems - Maximum independent set (MIS), Maximum clique, Minimum vertex cover, etc.

- Computing MIS: NP-hard
  - In general graphs, cannot approximate better than $n^{1-\epsilon}$
  - In intersection graphs, $(1 + \epsilon)$ approximations known

Distance based Graphs

☆ P - point set in plane

☆ Unit distance graph



☆ V - point set P

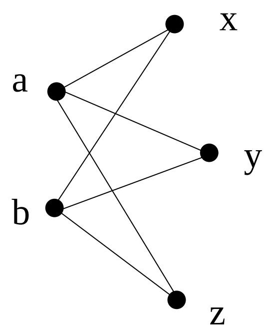
☆ $(i, j) \in E$ if $d(i, j) = 1$

Unit Distance Graph

- V - point set P
- $(i, j) \in E$ if $d(i, j) = 1$

- Maximum number of edges? (Erdos)
  - Over all possible n-point set

- $O(n^{3/2})$ edges
  - Forbidden $K_{2,3}$



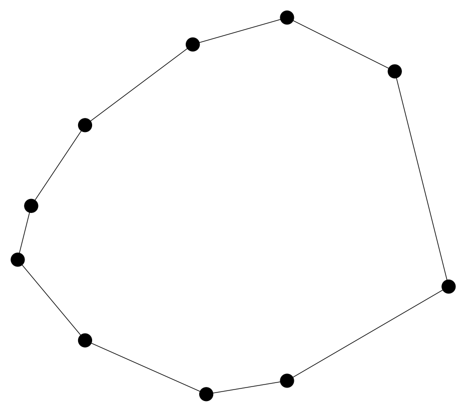
- $O(n^{4/3})$ edges
  - Crossing Lemma, Cuttings, Arrangement of Circles

Unit Distance Graph

- Upper bound
  - $O(n^{4/3})$ edges
- Lower bound
  - $\Omega(n^{1+\frac{c}{\log \log n}})$ [Erdos]
- Conjecture: Lower bound is tight

Unit Distance Graph - Convex Point Set

- ☆ Convex Point Set



- ☆ Upper bound: $O(n \log n)$ edges

- ☆ Lower bound: $2n - 7$ edges

- ☆ Conjecture: Lower bound is tight ($2n$ edges)

Questions

Questions