Fixed-Parameter Algorithms

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Outline

Introduction – The paradigm and motivation

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  The running time is analyzed in terms of *both* the input size and the parameter.
  Confine the combinatorial explosion to parameters that are likely to be small.
  The aim is to design algorithms that take $O(f(k)|x|^{O(1)})$ time where f is a (moderately) exponential function.

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- Active area of research, there are sessions on this topic in major conferences, there are specialized conferences – IPEC (international symposium on parameterized and exact computation; 2010 event in Chennai).

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- A parameterized language (YES instances of a decision problem) is *fixed-parameter tractable* (FPT) if there is an algorithm that can decide for any input (x, k), whether it is in the language in time $f(k)|x|^{O(1)}$ time where f is any function of k;
- FPT algorithms (with moderately growing $f(k)$) are useful in practice when the parameter k is small; and there are areas where small parameters capture most practical instances.

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   If $n > R(3, k)$ (the Ramsey number), then $G \in L$
   else ($n \leq R(3, k) \leq k^2$) check whether G has an independent set of size k by trying all subsets of size at most k.

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Domination number is a contraction closed bidimensional parameter, and hence treewidth of a planar graph with a k-dominating set is $O(\sqrt{k})$, apply dynamic programming.

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4. For a fixed k, the following families are minor closed.
   $L_k = \{Graphs\ having\ at\ most\ k\ sized\ vertex\ covers(VC)\}$
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   and hence by the above two theorems, checking whether a graph has a VC or FVS of size at most k is FPT.
5. However the $f()$ had a HUGE towers of exponents and the proof of GMT is also non constructive!

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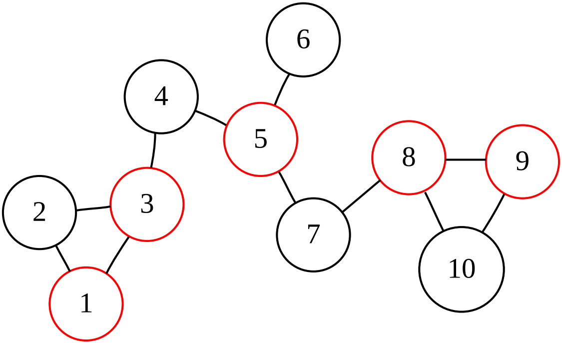
4. Downey-Fellows developed hardness theory ($W[1]$, $W[2]$-complete problems) and opened up the area (in late eighties, early nineties).

Outline

Bounded Search Trees – example Vertex Cover

Given a graph $G = (V, E)$, and a parameter k, does G have a vertex cover of size at most k?



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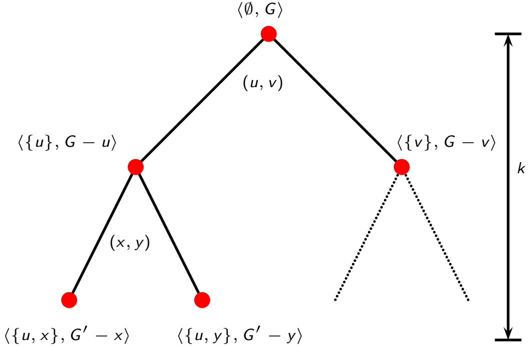
Basic Idea: Given *any* edge (u, v) either u or v is in the solution.



- no. of nodes $\leq 2^k$
- time spent at each node $= O(m)$
- total time taken $= O(2^k \cdot m)$

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   If every vertex has degree at most 1, solve in polynomial time. $T(k) \leq T(k-1) + T(k-2)$ Fibonacci recurrence on k that results in $O((1.618)^k m)$.
2. Can be improved by branching on larger structures and doing a lot of case analyses; the current best is $O(1.27^k + kn)$.

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- Time complexity: $O(n^k m)$.

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Lemma (Erdos and Posa)

An undirected graph on n vertices with minimum degree 3 has a cycle of length $O(\log n)$.

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- Since $(\log n)^k \leq k^{2k} + n$, we have an FPT algorithm, where $f(k) = k^{2k}$.

Towards a better $f(k)$

- A generalization of Erdos and Posa:

Lemma

Any graph with minimum degree 3 *and FVS of size at most* $\sqrt{n/2}$*, has a cycle of length at most* 6*.*

  Using this, one can get a bound of $O^*((\log k)^k)$.

- Improving to $O^*(c^k)$, will be done later.

Kernalization - A formal way of analysing *preprocessing strategies*

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3. Theorem (easy): A parameterized problem is kernalizable implies it is in FPT.
4. Converse is also true: A parameterized problem is FPT implies it is kernalizable; but the kernel size will be exponential in k.

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6. Now we can do branching on G'. Resulting run time is $O(kn + (1.28)^k)$.

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3. *Famous Open problems:* Polynomial sized ($k^{O(1)}$) kernel for
   - 3.1 Directed feedback vertex set?
   - 3.2 Odd cycle transversal (set of vertices whose removal results in a bipartite graph)?

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4. Now, very recently (STOC 08, ICALP 08, 09) there is a machinery (composition) to show NON existence of polynomial sized kernal (unless $PH = \Sigma^3$)
5. k-path, k-connected vertex cover are some problems that don't have polynomial sized kernel under this hypothesis.

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4. Label the vertices $v_1, \ldots v_n$, $\{v_1, ...v_{k+1}\}$ is a $k+1$-sized solution for $G_{k+1} = G[\{v_1, ...v_{k+1}\}]$. Apply the compression step, if this can not be compressed, G_{k+1} has no k sized solution and G also has no k-sized solution.

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5. If G_{k+1} has a k-sized solution S, then $S \cup \{v_{k+2}\}$ is a $(k+1)$-sized solution for G_{k+2} and continue like this.
6. Overall time is $O((n-k) * \textit{time for compression step})$.

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- for every subset $X \subseteq S$, $|X| \leq k$, we will check whether there is an FVS of G of size at most k containing all of X and none of $S - X$.
- Since there are at most $2^{k+1} - 1$ such subsets X, the problem is FPT if we show it FPT for a fixed X.

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- We look for an FVS of $G - X$ of size at most $k - |X|$ containing no vertex of $S - X$.
- X can be deleted from G.

The Compression problem for FVS

Given $G = (V, E)$, a subset S, which is a FVS of size $k+1$, a subset X of S of size at most k, find a FVS of size at most $k - |X|$ from $V - S$.

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2. $G[S - X]$ has at most $k + 1 - |X|$ vertices and hence at most that many components

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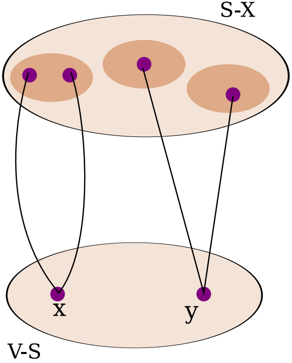

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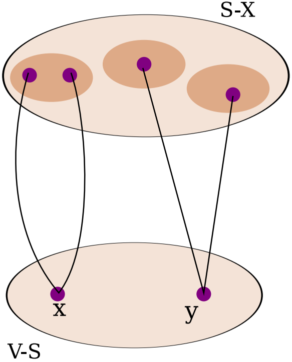

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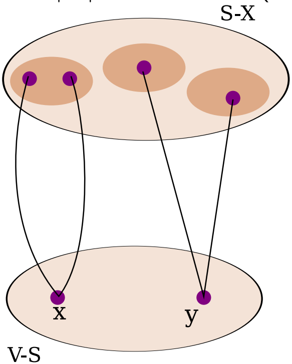
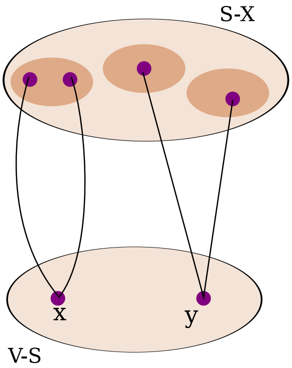


The Algorithm:

1. Let x be a vertex of degree at most 1 in $G[V-S]$. It has at least *two* neighbors in $S-X$.
2. If both neighbors are in the same component of $G[S-X]$, then include x in FVS (forced).



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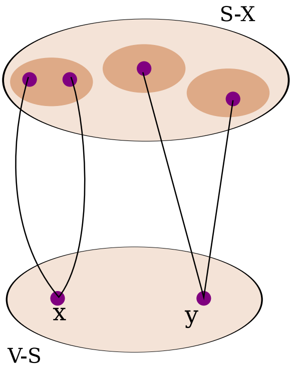

1. Else (let y be a vertex whose neighbors are in at least two components of $G[S - X]$), we branch
    1.1 by picking y in FVS, in which case k drops by 1 in the recursion, or
    1.2 by not picking y in which case y is added to $S - X$ reducing the number of components of $S - X$ by 1.

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3. Overall runtime (with $2^{k+1} - 1$ choices for X) is $4^k 2^k \text{poly}(n)$.
4. With a careful analysis, this turns out to be $O(5^k n^{O(1)})$.
5. This is the best known bound, the open problem is to improve 5^k.

More on Iterated Compression

Several recent results were shown FPT using iterated compression

1. Directed Feedback Vertex Set (STOC 08, JACM 2009)
2. Within k clauses from 2SAT (ICALP 08)
3. Cochromatic Number in perfect graphs
4. Odd Cycle Transversal (the first one, in ORL)

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  5. Can be derandomized using perfect hash families

More on Color Coding

1. Can find k-path, k-cycle, k-tree, subgraphs of bounded treewidth with k vertices all in FPT time.
2. Recently applied to get a $2^{O(\sqrt{k}\log k)} + n^{O(1)}$ algorithm for finding Feedback Arc Set in tournaments (ICALP 2009).

Summary of Algorithmic Techniques

1. Graph Minor Theory, MSO, Treewidth machinery (mainly for Classification)
2. Bounded Search Trees
3. Reduction to Kernel
4. Iterated Compression
5. Color Coding

Outline

The hardness theory and reductions

1. Parameterized reductions – An algorithm that reduces (x, k) of problem A to equivalent (x', k') of problem B
   - time allowed is $f(k)|x|^c$ time,
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Approximation and FPT

1. For all maximization problems in MAXSNP, their corresponding decision question is in FPT (with the solution size as the parameter)
2. There are easy to approximate problems whose decision versions are W-hard (rectangle stabbing) and
3. There are FPT problems (k-path, odd cycle traversal) whose optimization versions are hard to approximate.
4. EPTAS implies the corresponding decision version is FPT.
5. Last word not out yet!

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6. FPT Approximation for W-hard problems

Outline

Conclusions

1. Reasonably young, at the same time, has reasonably rich theory
2. Well-developed techniques – some simple, some use heavy machinery
3. Lots of open problems, combinatorial results
4. Practical applications in computational biology, optimization

References

1. Invitation to Fixed-Parameter Algorithms – Rolf Niedermeir (Oxford UP 2006)
2. Parameterized Complexity – Rod Downey and Mike Fellows (Springer 1999)
3. Parameterized Complexity Theory – Jörg Flum and Martin Grohe (Springer 2006)
4. Proceedings of IWPEC, and other conferences

Thank You!