Geometric data structures

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Scope of the lecture

- **Binary search trees and 2-d range trees**
  We consider 1-d and 2-d range queries for point sets.

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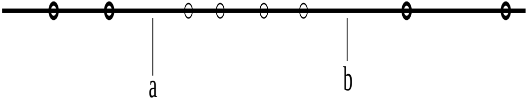
- **Paradigm of Sweep algorithms**
  For reporting intersections of line segments, and for computihg visible regions.

1-dimensional Range searching



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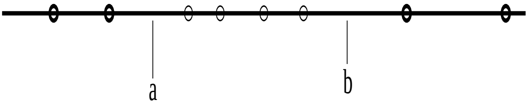

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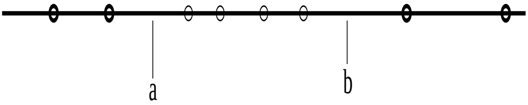

- Problem: Given a set P of n points $\{p_1, p_2, \cdots, p_n\}$ on the real line, report points of P that lie in the range $[a, b]$, $a \leq b$.
- Using binary search on an array we can answer such a query in $O(\log n + k)$ time where k is the number of points of P in $[a, b]$.
- However, when we permit insertion or deletion of points, we cannot use an array answering queries so efficiently.

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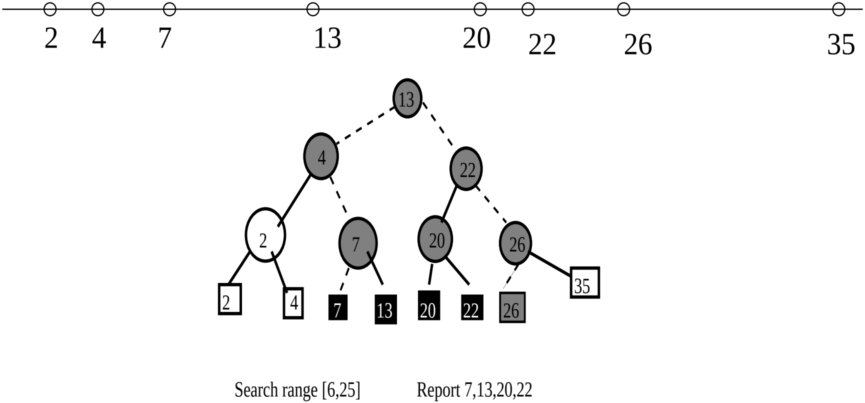

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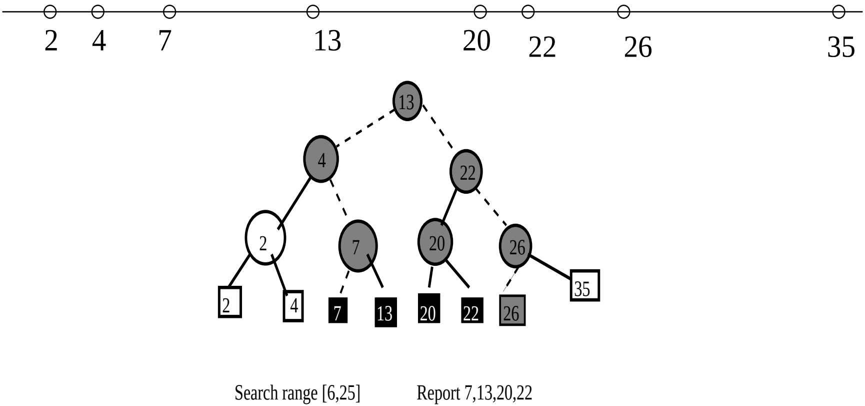
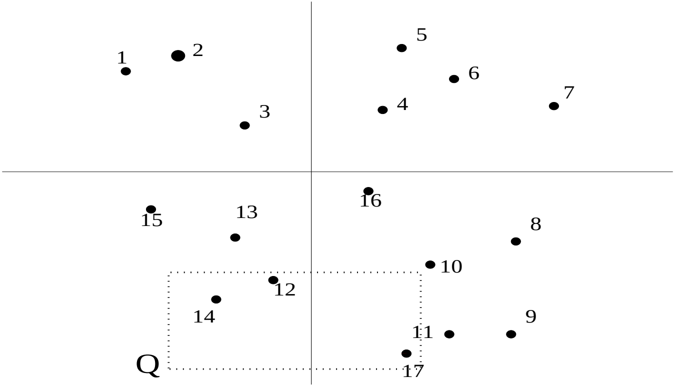


- We use a *binary leaf search tree* where leaf nodes store the points on the line, sorted by x-coordinates.
- Each internal node stores the x-coordinate of the rightmost point in its left subtree for guiding search.

2-dimensional Range Searching



- Problem: Given a set P of n points in the plane, report points inside a query rectangle Q whose sides are parallel to the axes.

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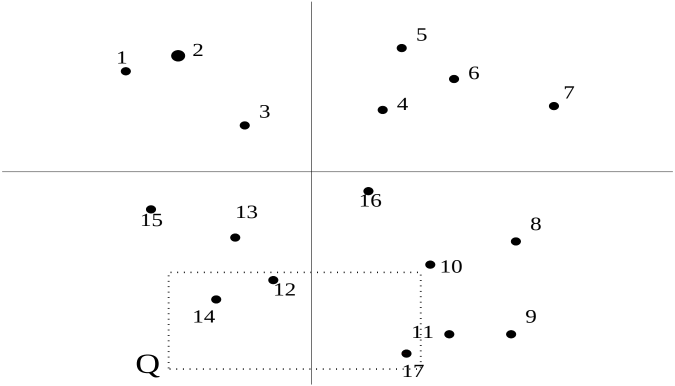

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- Here, the points inside R are 14, 12 and 17.

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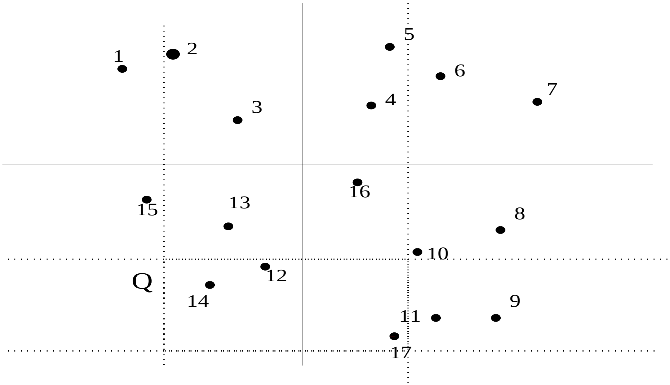

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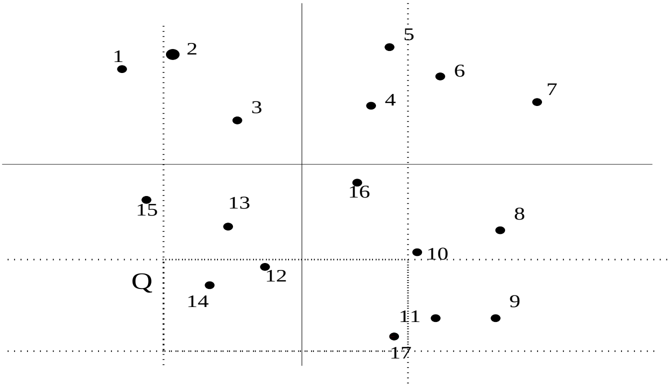

- Using two 1-d range queries, one along each axis, solves the 2-d range query.
- The cost incurred may exceed the actual output size of the 2-d range query.

Range searching with range trees and Kd-trees

- Given a set S of n points in the plane, we can construct a *2d-range tree* in $O(n \log n)$ time and space, so that rectangle queries can be executed in $O(\log^2 n + k)$ time.

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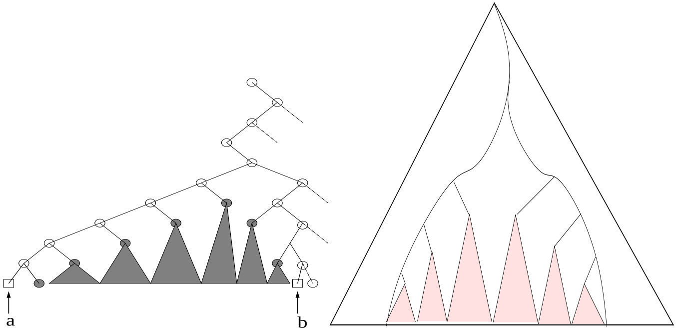
- The query time can be improved to $O(\log n + k)$ using the technique of *fractional cascading*.
- Given a set S of n points in the plane, we can construct a Kd-tree in $O(n \log n)$ time and $O(n)$ space, so that *rectangle queries* can be executed in $O(\sqrt{n} + k)$ time. Here, the number of points in the query rectangle is k.

Range searching in the plane using range trees



Given a 2-d rectangle query $[a, b]X[c, d]$, we can identify subtrees whose leaf nodes are in the range $[a, b]$ along the X-direction.

Only a subset of these leaf nodes lie in the range $[c, d]$ along the Y-direction.

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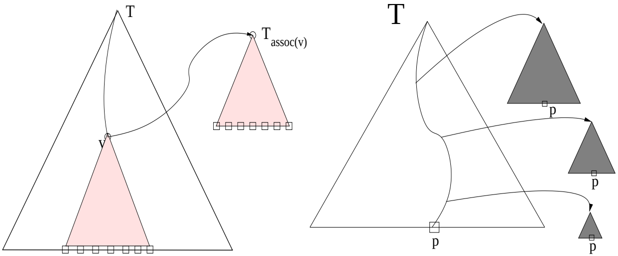
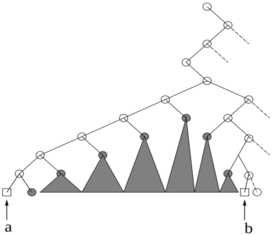


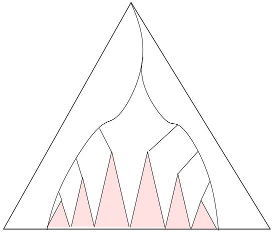
$T_{assoc(v)}$ is a binary search tree on y-coordinates for points in the leaf nodes of the subtree tooted at v in the tree T.

The point p is duplicated in $T_{assoc(v)}$ for each v on the search path for p in tree T.

The total space requirement is therefore $O(n \log n)$.

Range searching in the plane using range trees

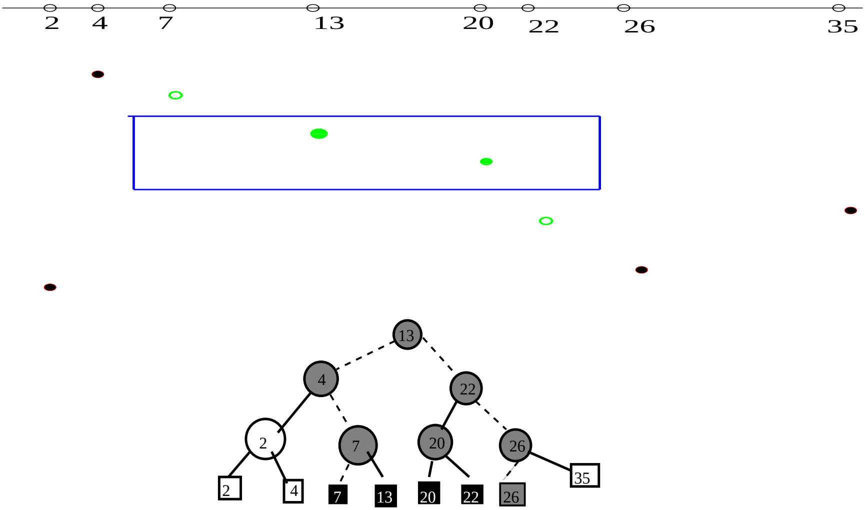




We perform 1-d range queries with the y-range $[c, d]$ in each of the subtrees adjacent to the left and right search paths within the x-range $[a, b]$ in the tree T.

Since the search path is $O(\log n)$ in size, and each y-range query requires $O(\log n)$ time, the total cost of searching is $O(\log^2 n)$. The reporting cost is $O(k)$ where k points lie in the query rectangle.

2-RANGE TREE SEARCHING



Search range [6,25]   Report 7,13,20,22

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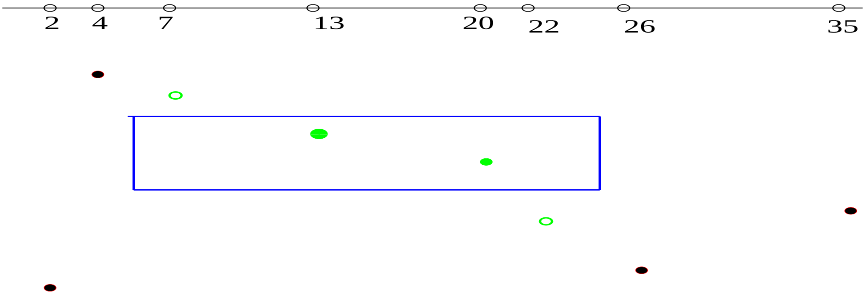

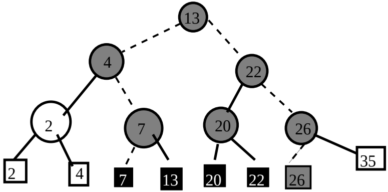

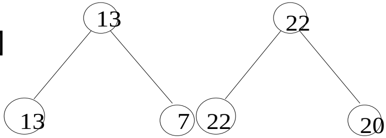

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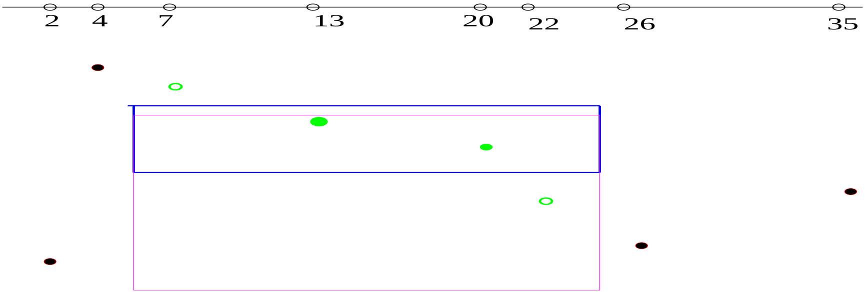

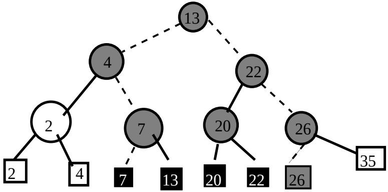

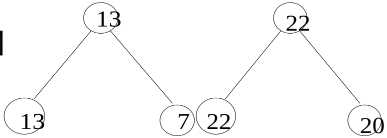

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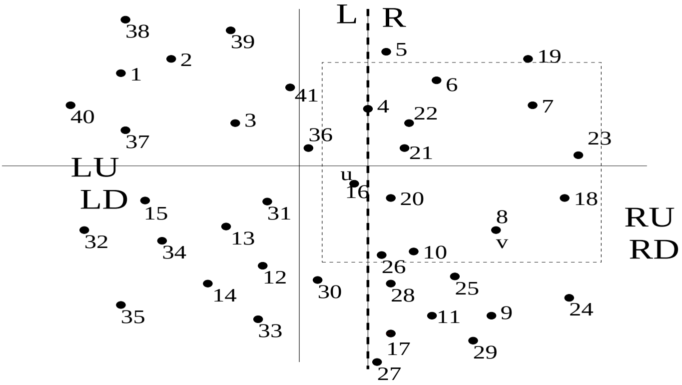
Partition by the median of x-coordinates



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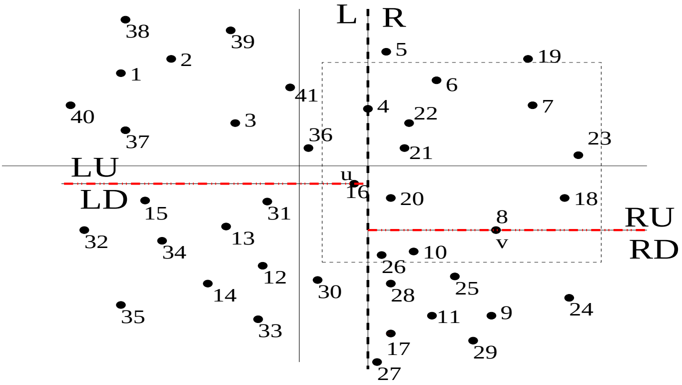

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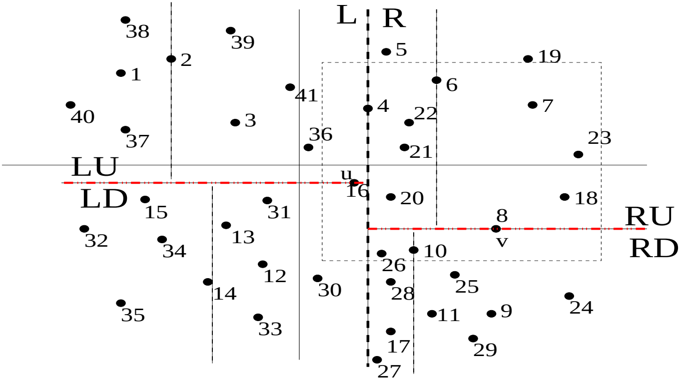

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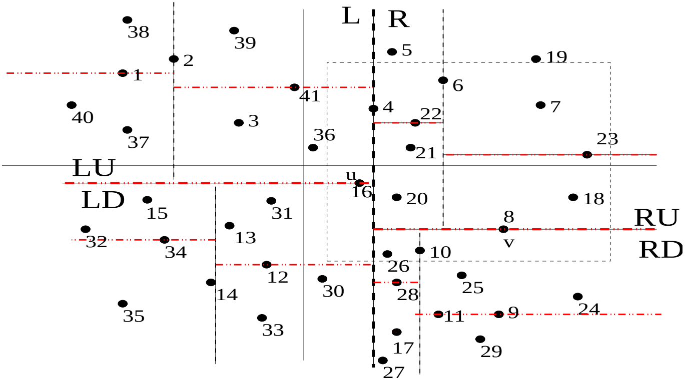
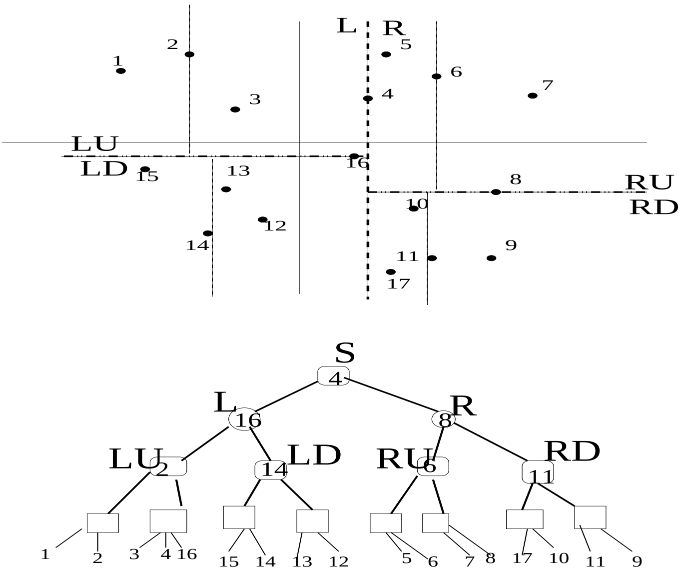
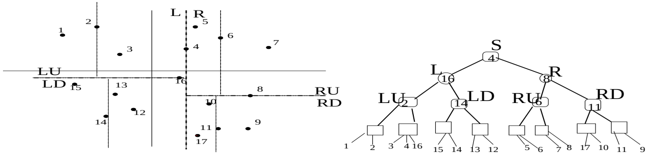


2-dimensional range searching using Kd-trees



Description of the Kd-tree



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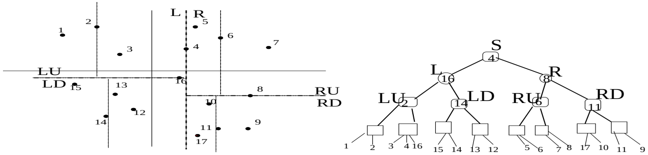

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- The point r stored in the root vertex T splits the set S into two roughly equal sized sets L and R using the median x-cooordinate $xmedian(S)$ of points in S, so that all points in L (R) have abscissae less than or equal to (strictly greater than) $xmedian(S)$.

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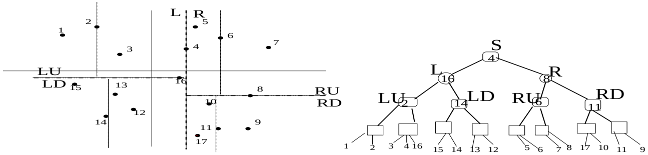

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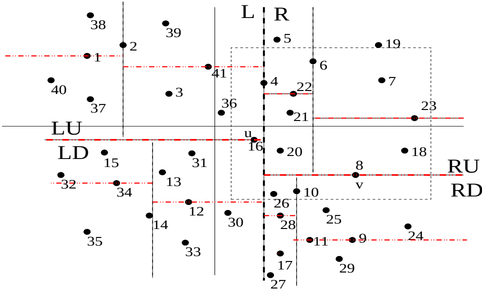
- The entire plane is called the $region(r)$.

Answering rectangle queries



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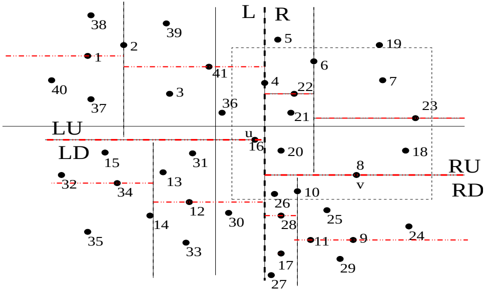

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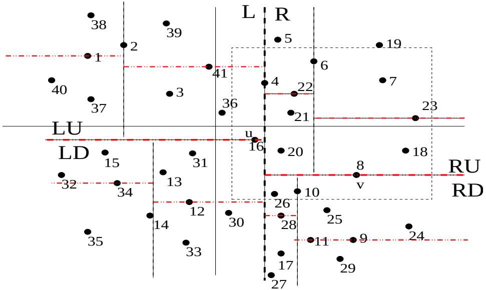

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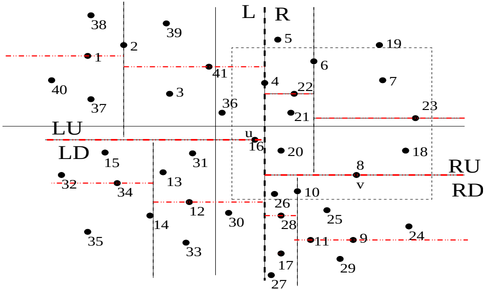

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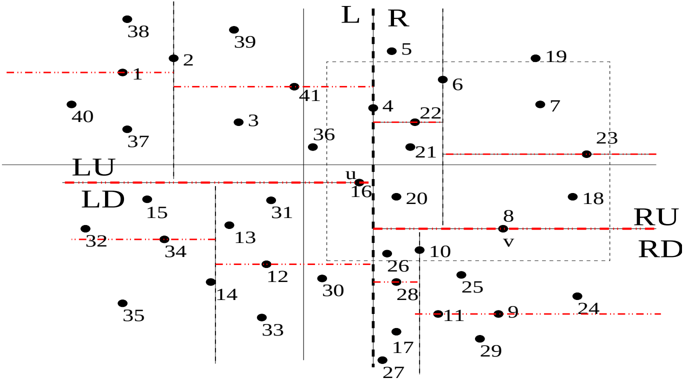
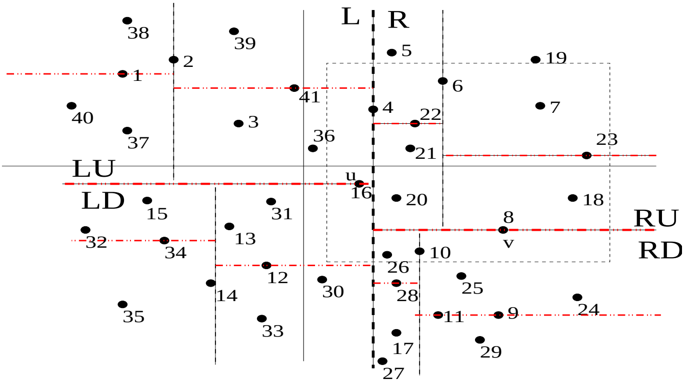
- If R overlaps $region(p)$ then we check whether R also overlaps the two regions of the children of the node N.

2-DIMENSIONAL RANGE SEARCHING: KD-TREES



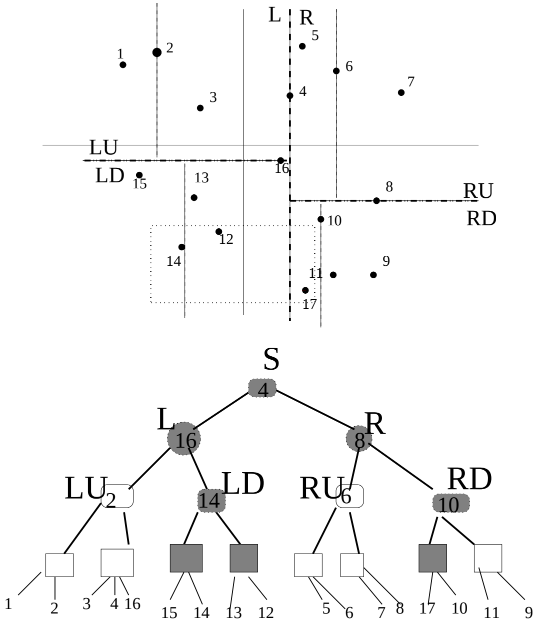
- The set L (R) is split into two roughly equal sized subsets LU and LD (RU and RD), using point u (v) that has the median y-coordinate in the set L (R), and including u in LU (RU).

2-dimensional Range Searching: Kd-trees



- The set L (R) is split into two roughly equal sized subsets LU and LD (RU and RD), using point u (v) that has the median y-coordinate in the set L (R), and including u in LU (RU).
- The entire halfplane containing set L (R) is called the $region(u)$ ($region(v)$).

Nodes traversed in the Kd-tree



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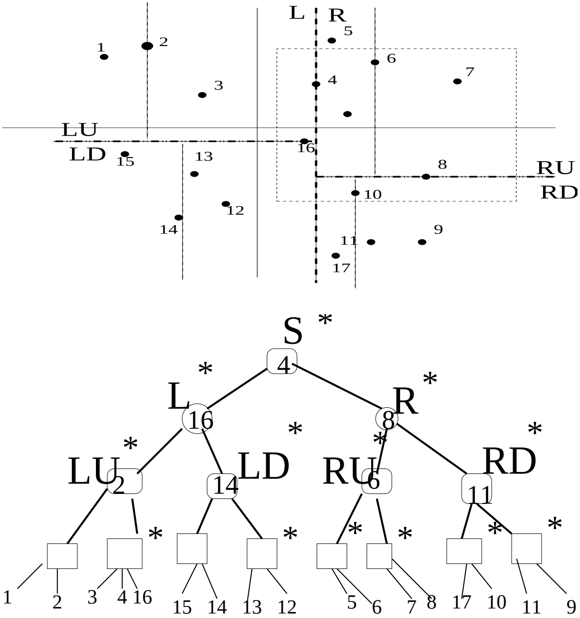

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- This cost is borne for all leaf level regions intersected by R.

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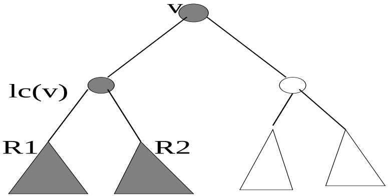
- Any vertical line intersecting S can intersect either L or R but not both, but it can meet both RU and RD (LU and LD).
- Any horizontal line intersecting R can intersect either RU or RD but not both, but it can meet both children of RU (RD).

Time complexity of rectangle queries for Kd-trees



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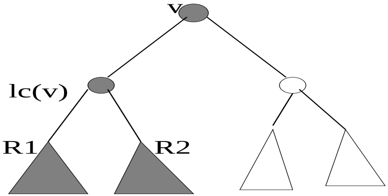

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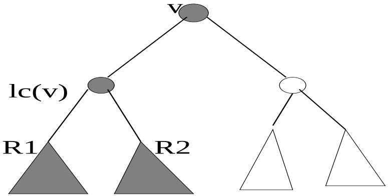

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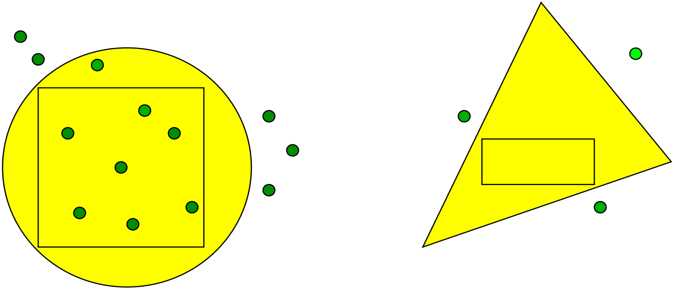
- The total cost of reporting k points in R is therefore $O(\sqrt(n) + k)$.

More general queries



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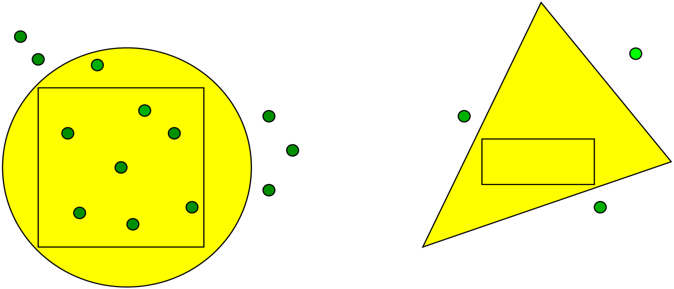

General Queries:

- Triangles can be used to simulate polygonal shapes with straight edges.
- Circles cannot be simulated by triangles either.

Triangle queries

- Using $O(n^2)$ space and time for preprocessing, triangle queries can be reported in $O(\log^2 n + k))$ time, where k is the number of points inside the query triangle.

Goswami, Das and Nandy: Comput. Geom. Th. and Appl. 29 (2004) pp. 163-175.

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- For counting the number k of points inside a query triangle, worst-case optimal $O(\log n)$ time suffices.

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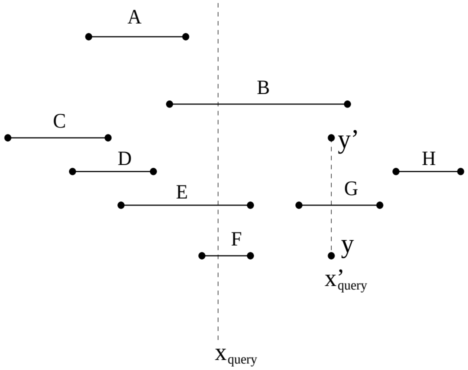
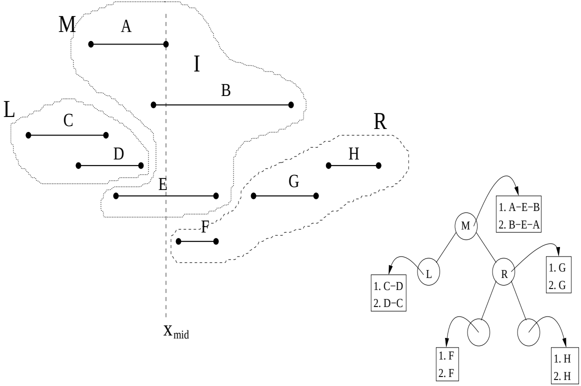
Finding intervals containing a query point



Simpler queries ask for reporting all intervals intersecting the vertical line $X = x_{query}$.

More difficult queries ask for reporting all intervals intersecting a vertical segment joining (x'_{query}, y) and (x'_{query}, y').

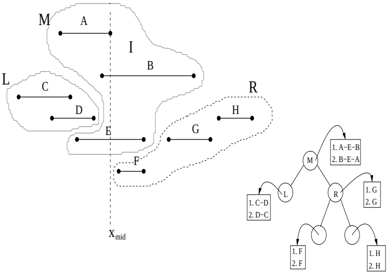
Computing the interval tree



The set M has intervals intersecting the vertical line $X = x_{mid}$, where x_{mid} is the median of the x-coordinates of the $2n$ endpoints.

The root node has intervals M sorted in two independent orders (i) by right end points (B-E-A), and (ii) left end points (A-E-B).

Answering queries using an interval tree



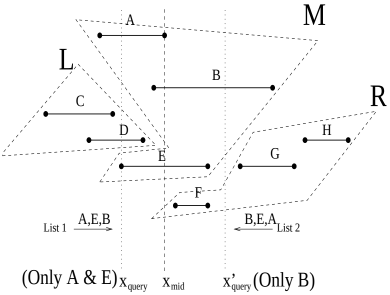
The set L and R have at most n endpoints each.

So they have at most $\frac{n}{2}$ intervals each.

Clearly, the cost of (recursively) building the interval tree is $O(n \log n)$.

The space required is linear.

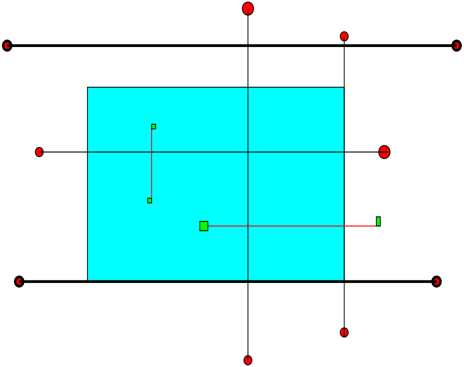
Answering queries using an interval tree

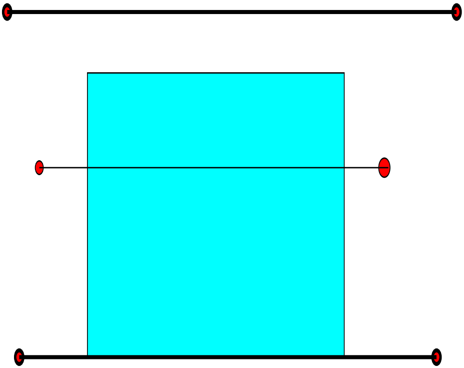


For $x_{query} < x_{mid}$, we do not traverse subtree for subset R.

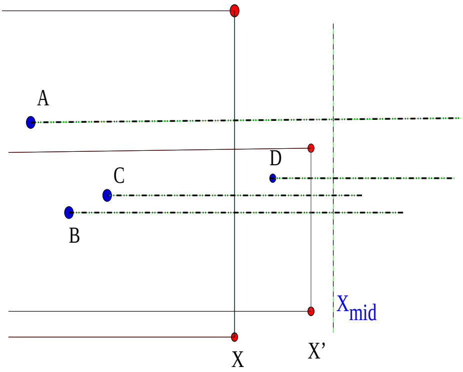
For $x'_{query} > x_{mid}$, we do not traverse subtree for subset L.

Clearly, the cost of reporting the k intervals is $O(\log n + k)$.





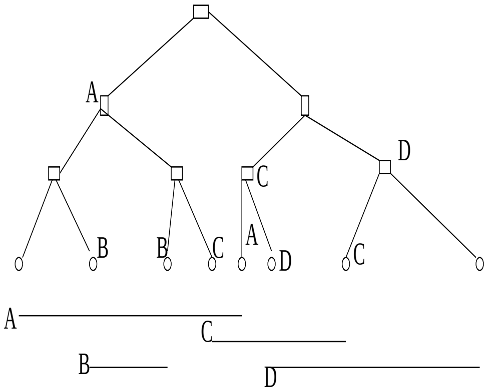
The problem is to report all (horizontal) segments that cut across the query rectangle or include an entire (top/bottom) bounding edge.



Use an interval tree of all the horizontal segments and the right bounding edge of the query rectangle like X or X'.

Use the rectangle query for vertical segment X and find points A, B and C in the rectangle with left edge at minus infinity. For X', report B, C and D, similarly.

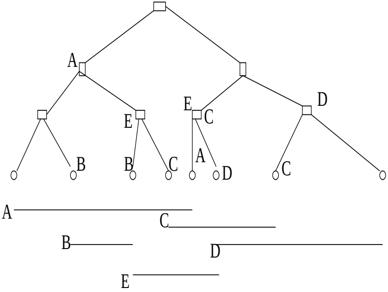
Introducing the segment tree



For an interval which spans the entire range $inv(v)$, we mark only internal node v in the segment tree, and not any descendant of v.

We never mark any ancestor of a marked node.

Representing intervals in the segment tree

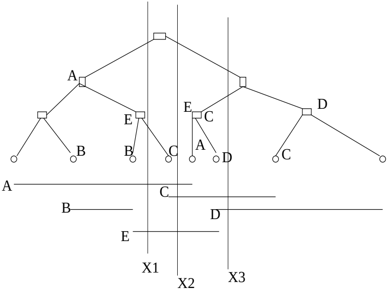


At each level, at most two internal nodes are marked for any given interval.

Along a root to leaf path an interval is stored only once.

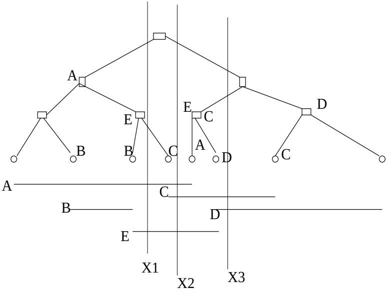
The space requirement is therefore $O(n \log n)$.

Reporting intervals containing a given query point



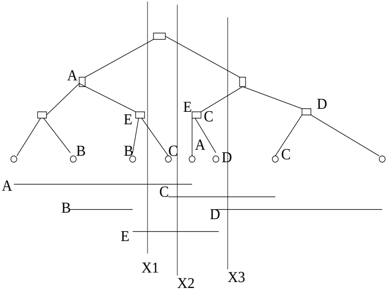
- Search the path in the tree reaching the leaf for the given query point.

Reporting intervals containing a given query point



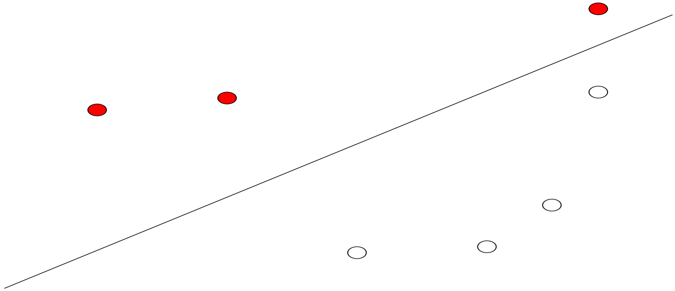
- Search the path in the tree reaching the leaf for the given query point.
- Report all intervals that appear stored on the search path.

Reporting intervals containing a given query point

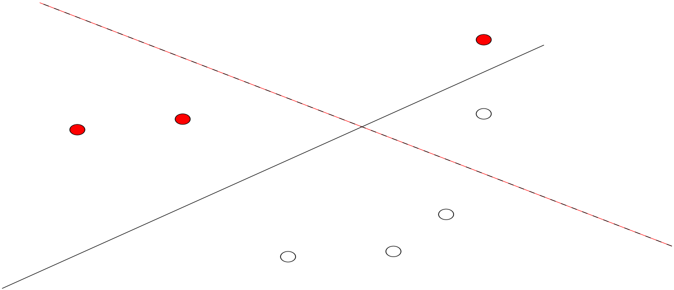


- Search the path in the tree reaching the leaf for the given query point.
- Report all intervals that appear stored on the search path.
- If k intervals contain the query point then the cost incurred is $O(\log n + k)$.

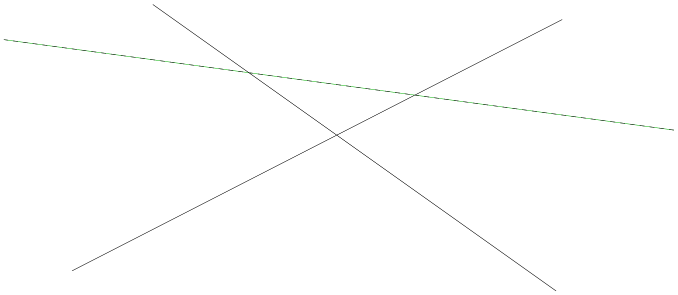
Halfplanar range queries



Halfplanar range queries



Halfplanar range queries using simulataneous bisectors



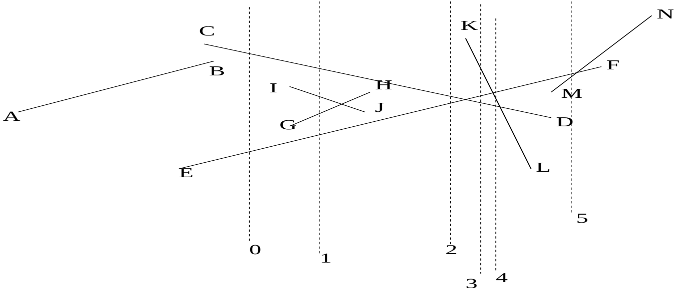
$T(n) \leq 3T(n/4) + c \log n$
OR
$T(n) \leq T(n/2) + T(n/4) + c \log n$

Halfplanar range queries

- Using $O(n \log n)$ time for preprocessing, *halfplanar* range queries can be reported in $O(n^{0.695} + k)$ time, where k is the number of points inside the query triangle.

Edelsbrunner and Welzl: Info. Proc. Lett. 23 (1986) pp. 289-293.

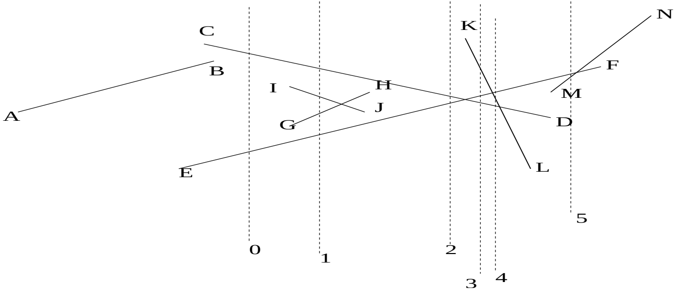
Reporting segments intersections



Problem: Given a set S of n line segments in the plane, report all intersections between the segments.

- Check all pairs in $O(n^2)$ time.

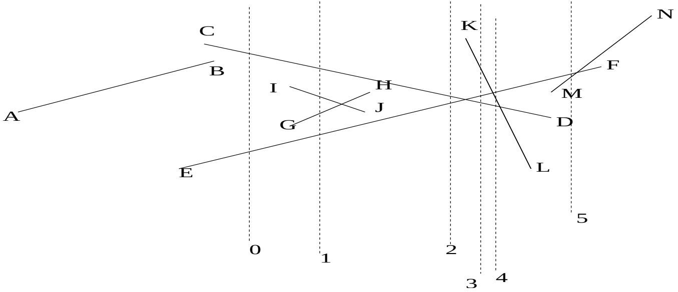
Reporting segments intersections



Problem: Given a set S of n line segments in the plane, report all intersections between the segments.

- Check all pairs in $O(n^2)$ time.
- A vertical line just before any intersection meets intersecting segments in an empty, intersection free segment.

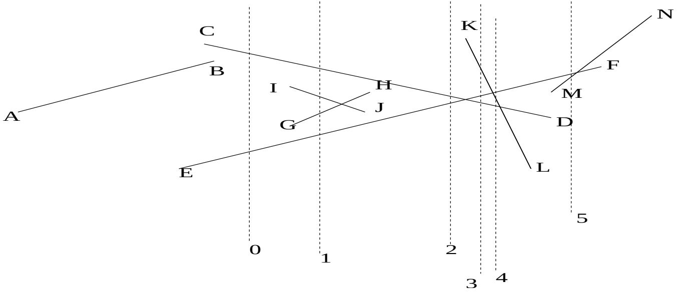
Reporting segments intersections



Problem: Given a set S of n line segments in the plane, report all intersections between the segments.

- Check all pairs in $O(n^2)$ time.
- A vertical line just before any intersection meets intersecting segments in an empty, intersection free segment.
- Detect intersections by checking consecutive pairs of segments along a vertical line.

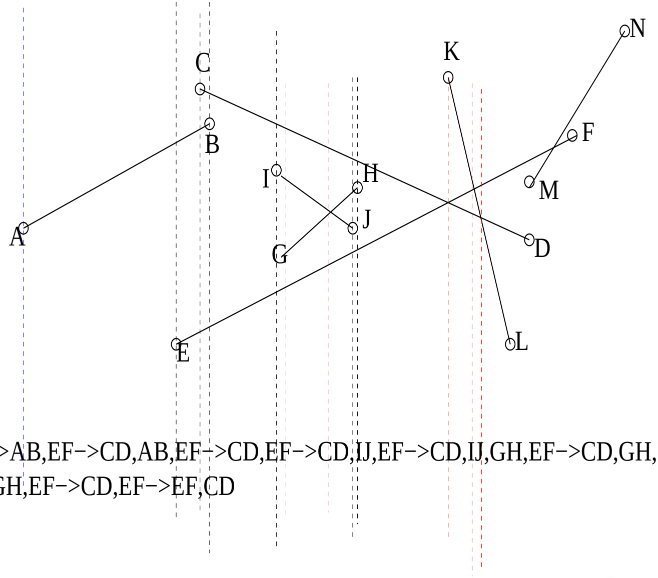
Reporting segments intersections



Problem: Given a set S of n line segments in the plane, report all intersections between the segments.

- Check all pairs in $O(n^2)$ time.
- A vertical line just before any intersection meets intersecting segments in an empty, intersection free segment.
- Detect intersections by checking consecutive pairs of segments along a vertical line.
- This way, each intersection point can be detected.

Sweeping steps: Endpoints and intersection points

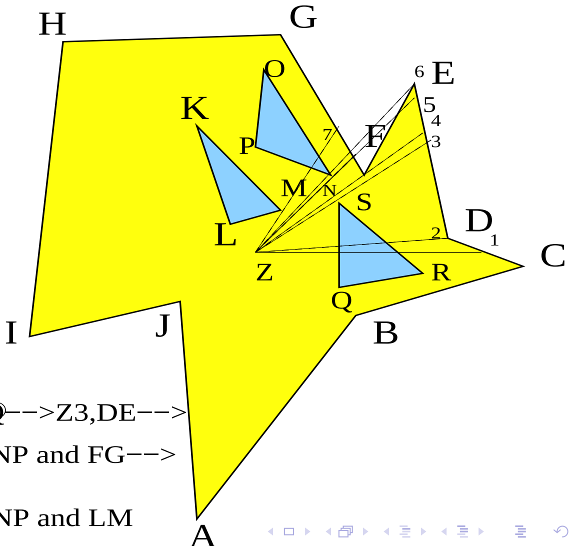


AB−>AB,EF−>CD,AB,EF−>CD,EF−>CD,IJ,EF−>CD,IJ,GH,EF−>CD,GH,IJ,EF
CD,GH,EF−>CD,EF−>EF,CD

SQ,SR,DC,1−−>SQ,SR,DE,2−−>DE,3−−

FG,FE,DE,4−−>NP,NO,FG,FE,DE,5−−>

NP,NO,FG,FE,DE,6−−>LM,MK,NP,NO,FG,7



Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

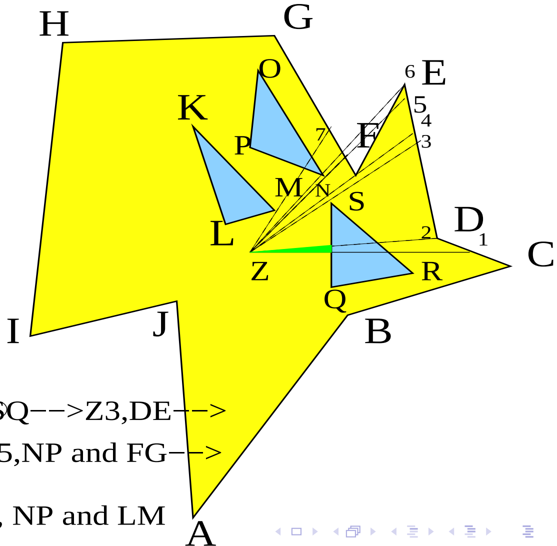
Z4,FG and DE−−>Z5,NP and FG−−>

Z6,NP−−>Z7, NP and LM

SQ,SR,DC,1−−>SQ,SR,DE,2−−>DE,3

FG,FE,DE,4−−>NP,NO,FG,FE,DE,5−−>

NP,NO,FG,FE,DE,6−−>LM,MK,NP,NO,FG,7



Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

Z4,FG and DE−−>Z5,NP and FG−−>

Z6,NP−−>Z7, NP and LM

SQ,SR,DC,1−−>SQ,SR,DE,2−−>DE,3

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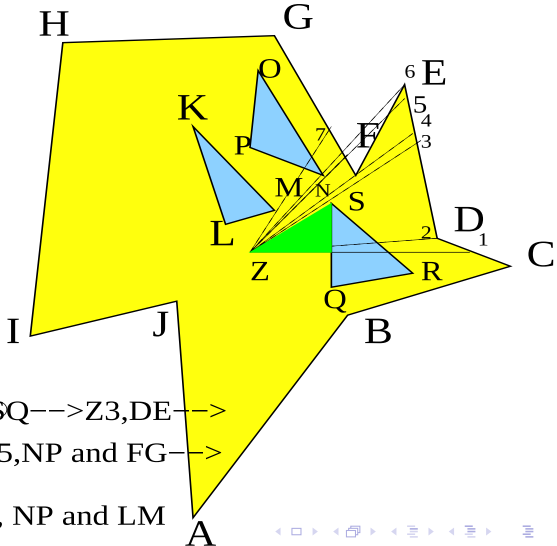

Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

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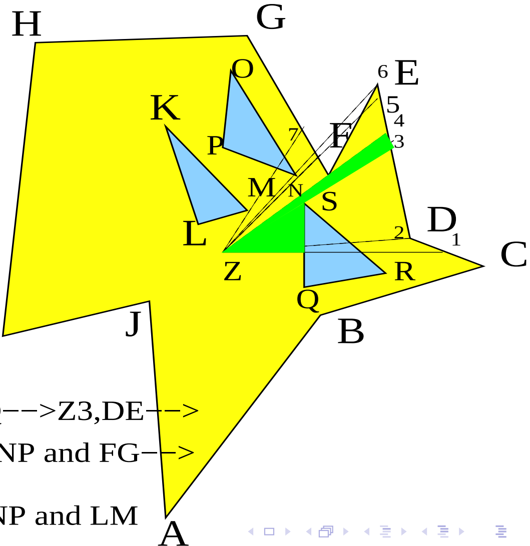

Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

Z4,FG and DE−−>Z5,NP and FG−−>

Z6,NP−−>Z7, NP and LM

SQ,SR,DC,1−−>SQ,SR,DE,2−−>DE,3

FG,FE,DE,4−−>NP,NO,FG,FE,DE,5−−>

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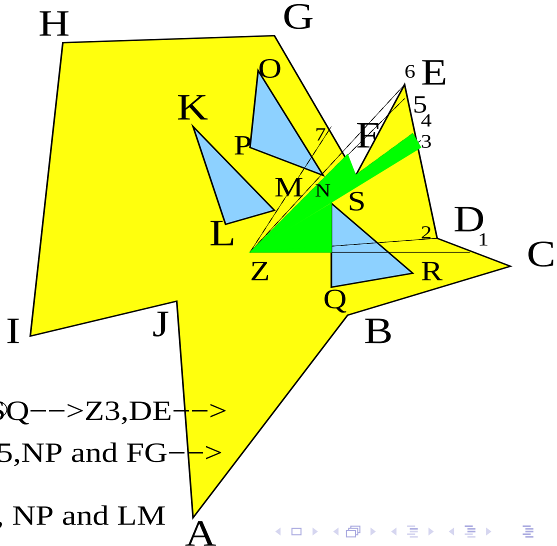


Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

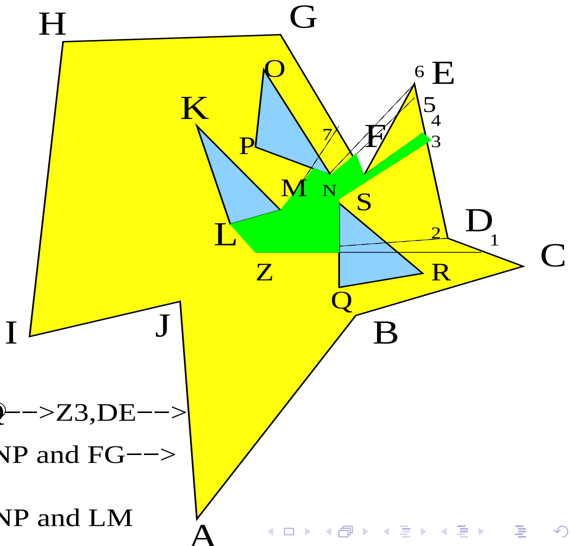
Z4,FG and DE−−>Z5,NP and FG−−>

Z6,NP−−>Z7, NP and LM

SQ,SR,DC,1−−>SQ,SR,DE,2−−>DE,3−-

FG,FE,DE,4−−>NP,NO,FG,FE,DE,5−−>

NP,NO,FG,FE,DE,6−−>LM,MK,NP,NO,FG,7



Z1 ,SQ−−>Z2,SQ−−>Z3,DE−−>

Z4,FG and DE−−>Z5,NP and FG−−>

Z6,NP−−>Z7, NP and LM

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