Labeled graphs and Digraphs: Theory and Applications

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Labeled graphs and Digraphs: Theory and Applications

- *Graph labelings*, where the *vertices* and *edges* are assigned, *real values* subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logico – mathematical).
- Graph labelings were first introduced in the mid sixties. In the intervening years, dozens of graph labeling techniques have been studied in over 1000 papers and is still getting embellished due to increasing number of application driven concepts.
- "Gallian, J. A., A dynamic survey of graph labeling, Electronic J. of Combinatorics, DS#6 , 2011, 1-246".

Labeled graphs and Digraphs: Theory and Applications

- *Labeled graphs* are becoming an increasingly useful family of *Mathematical Models* for a broad range of applications.
- Qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as Conflict resolution in social psychology], electrical circuit theory and energy crisis etc,..
- Quantitative labelings of graphs have led to quite intricate fields of application such as *Coding Theory* problems, including the design of good *Radar location codes, Synch-set codes*; *Missile guidance codes* and *convolution codes* with *optimal autocorrelation properties*.
- *Labeled graphs* have also been applied, in determining ambiguities in *X-Ray Crystallographic analysis, to Design Communication Network addressing Systems*, in determining *Optimal Circuit Layouts and* Radio-Astronomy., etc.

Most of the graph labeling problems have three ingredients:

(i) a set of number S from which the labels are chosen;

(ii) a rule that assigns a value to each edge;

(iii) a condition that these values must satisfy.

- Given a graph $G = (V, E)$, the set R of real numbers, a subset A of R and a commutative binary operation $*: R \times R \to R$, every vertex function $f: V(G) \rightarrow A$ induces an edge function $f^*: E(G) \to R$ such that $*f(uv) = f(u) * f(v)$, uv is an edge in G. In particular, f is said to be *integral* if its values lie in the set Z of integers.

GRAPH LABELINGS

- Problem: Minimize the value of the largest integer so assigned to any vertex of *G*, say $\theta(G)$. The Principal question which arises in the theory of graph labelings revolve around the relationship between $\theta(G)$ and *q*.

GRAPH LABELINGS

G is called a labeled graph if each edge $e=uv$ is given the value $f(uv) = f(u)*f(v)$, where * is a binary operation. In literature one can find * to be either *addition, multiplication, modulo addition or absolute difference, modulo subtraction or symmetric difference.*

In the absence of additional constraints, every graph can be labeled in infinitely many ways. Thus, utilization of numbered graph models requires imposition of additional constraints which characterize the problem being investigated.

GRAPH LABELINGS

The origins of the labeling go back to the Fourth Czechoslovakian *Symposium* on Combinatorics, Graphs, and Complexity, Smolenice, in 1963 where Gerhard Ringel proposed the following well-known conjecture.

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Ringel's Conjecture(RC)

The complete graph K_{2n+1} with $2n+1$ vertices can be decomposed into $2n+1$ subgraphs, each isomorphic to a given tree with n edges

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Given a graph $G = (V, E)$ with n edges and a mapping $\phi : V \rightarrow N$ (the set of nonnegative integers), consider the following conditions:

(a) $\phi(V) \subseteq \{0, 1, 2, \ldots, n\}$

(b) $\phi(V) \subseteq \{0, 1, 2, \ldots, 2n\}$

(c) $\phi(E) \subseteq \{1, 2, \ldots, n\}$

(d) $\phi(E) \subseteq \{x_1, x_2, \ldots, x_n\}$ where $x_i = i$ or $x_i = 2n+1-i$;

(e) There exists x such that either $\phi(u) < x \leq \phi(v)$ or $\phi(v) \leq x < \phi(u)$ whenever $\{u, v\} \in E$

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- α-labeling satisfies (a), (c) and (e).
- β-labeling (=graceful) satisfies (a) and (c).
- σ-labeling satisfies (b) and (c).
- ρ-labeling satisfies (b) and (d).

Among these α-labeling is the strongest and ρ-labeling is the weakest.

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- From the definition it immediately follows that,
- 1. The hierarchy of the labelings is, α-,β-,σ-, ρ-labelings, each labeling is at the same time is also a succeeding labeling of the given graph.
- 2. If there exists a α–valuation of a graph G, then G must be bipartite.
- 3. If there exists a β–valuation of a graph G with *m* vertices and *n* edges, then m-$n \leq 1$.

Prominent conjectures

- Kotzig's conjecture (KC): The complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs, each isomorphic to a given tree with n edges.
- Graceful tree conjecture (GTC): every tree has a graceful labeling.
- The ρ-labeling conjecture (ρC): Every tree has ρ-labeling.

Thus GTC implies KC which is equivalent to ρC which in turn implies RC.

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- By turning an edge in a complete graph K_n we mean the increase of both indices by one, so that from the edge $(v_i v_j)$ we obtain the edge $(v_{i+1} v_{j+1})$, the indices taken modulo n. By turning of a subgraph G in K_n we mean the simultaneous turning of all edges of G. A decomposition R of K_n is said to be cyclic, if R contains G, then it contains the graph obtained by turning G also.

- A tree T on n edges cyclically decomposes K_{2n+1} if there exists an injection g:V(T)→ Z_{2n+1} such that, for all distinct i,j in Z_{2n+1} there exists a unique k in Z_{2n+1} with the property that there is a pair of adjacent vertices u,v in T satisfying $\{i,j\} = \{g(u)+k, g(v)+k\}$

Theorems (Rosa)

Theorem: The complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs, each isomorphic to a graph G with n edges if and only if G has a ρ-labeling.

Theorem: If a graph G with n edges has an α-labeling, then there exists a decomposition of K_{2kn+1} into copies of G, for all k = 1, 2, ….

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Proved results

- GTC holds for trees of diameter up to 5.

- RC hold for any tree of diameter up to 7.

- Any tree with $\leq$ 27 vertices has graceful labeling.

- RC hold for any tree with $\leq$ 55 vertices.

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APPLICATIONS

- 1. Ambiguities in X-Ray crystallography

- Determination of Crystal structure from X-ray diffraction data has long been a concern of crystallographers. The ambiguities inherent in this procedure are now being understood.

- J.N. Franklin, ambiguities in the X-ray analysis of crystal structures, Acta Cryst., Vol. A 30, 698-702, Nov. 1974.
- G.S. Bloom, Numbered undirected graphs and their uses: A survey of unifying scientific and engineering concepts and its use in developing a theory of non-redundant homometric sets relating to some ambiguities in x-ray diffraction analysis, Ph. D., dissertation, Univ. of Southern California, Loss Angeles, 1975)

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APPLICATIONS

- 2. Communication Network Labeling

- In a small communication network, it might be useful to assign each user terminal a “node number” subject to the constraint that all connecting edges (communication links) receive distinct numbers. In this way, the numbers of any two communicating terminals automatically specify the link number of the connecting path; and conversely; the path number uniquely specifies the pair of user terminals which it interconnects.

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Applications

- Properties of a potential numbering system for such a networks have been explored under the guise of gracefully numbered graphs. That is, the properties of graceful graphs provide design parameters for an appropriate communication network. For example, the maximum number of links in a network with m transmission centers can be shown to be asymptotically limited to not more than 2/3 of the possible links when m is large.

APPLICATIONS

- 3. Construction of polygons of same internal angle and distinct sides:

- Using a labeling of a cycle C_{2n+1}, we can construct a polygon P_{4n+2} with $4n+2$ sides such that all the itnernal angles are equal and lengths of the sides are distinct.

- S.M. Hegde and Sudhakar Shetty, Strongly indexable graphs and applications , Discrete Mathematics, 309 (2009) 6160-6168.

APPLICATIONS

- Ambiguities in X-ray crystallography
- Sometimes it happen that distinct crystal structures will produce identical X-ray diffraction patterns. These inherent ambiguities in x-ray analysis of crystal structures have been studied by Piccard, Franklin and Bloom.

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APPLICATIONS

- In some cases the same diffraction information may corresponds to more than one structure. This problem is mathematically equivalent to determining all labelings of the appropriate graphs which produce a prespecified set of edge numbers

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APPLICATIONS

- Franklin studied finite sets of points that would give same diffraction pattern. He called these sets as strictly homometric (or more simply, homometric) . He discovered a construction to produce families of homometric sets.
- Conditions for a pair of sets to be homometric: Two sets R and S are said to be homometric if $S \neq \pm R + c$ and $D(S) = D(R)$.

APPLICATIONS

- Consider the sets
- R = {0,1,3,4,5,7,9,10,12} and S = {0,2,3,4,6,7,9,11,12}.
- Then, D(R)= { $|a-b|$: a, b $\in$ R} = { $|c-d|$:c, d $\in$ S} = D(S)
- =
- {1,1,1,1,2,2,2,2,2,3,3,3,3,3,4,4,4,4,5,5,5,5,6,6,6,7,7,7,8,8,9,9,9,10,11,12}

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- Display of each of these two 9-element homometric sets is graphically done with complete graph with 9 vertices. Thus homometric sets can thus be defined as distinct sets of vertex numbers on complete graphs that generate identical edge numbers.

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APPLICATIONS

- An interesting development in the research of homometric sets has occurred in the year 1975. Piccard in 1939 presented a theorem which crystallographers immediately accepted for its narrowing of the necessary scope of their investigations into diffraction pattern ambiguities. It was believed that Piccard had proved "if all elements in a difference set are distinct, there is a unique set that would generate it". i.e., no pair of homometric sets was believed to exist with a difference set comprised of distinct elements

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APPLICATIONS

- As one can see Franklins constructions of homometric sets do not violate this condition since his difference sets always include repeated elements.

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APPLICATIONS

- In 1975, Bloom and Golomb proved that this theorem was not true by producing many pairs of non redundant homometric sets.

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APPLICATIONS

- R = {0,1,4,10,12,17}   and   S = {0,1,8,11,13,17}
- D(R) = {1,2,3,4,5,6,7,8,9,10,11,12,13,16,17} = D(S)

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APPLICATIONS

- . To date no other homometric Golomb Ruler pairs have been found. This discovery prompted many questions as to whether other such counter examples exist, whether this is a minimum counter example, and whether such counter examples relate to other “special” labelings of the complete graphs.

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APPLICATIONS

- Bloom and Golomb, have generalized the original counter example to a two-parameter family of nonredundant homometric rulers for which
- R= {$0, u, u+v, 4u+2v, 6u+2v, 8u+3v$} and
- S = { $0, u, 5u+v, 5u+2v, 7u+2v, 8u+3v$}

APPLICATIONS

- The original counter example is obtained by setting u=1 and v=3. Not all choices of u and v result in sets of distinct differences, but as a concrete example, if v is kept fixed at 3, each positive integer value of u generates a different counter example to "Piccard's Theorem".

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Remarks

- It was proved that the complete graph with 5 or more vertices cannot be gracefully labeled. Then the question " How well can we label large graphs on *n* vertices ?" led to two theoretical directions. Each of these has practical applications.
- The direction taken by labeling the complete graph as well as possible led to relaxing the constraint on the largest allowable vertex number which in turn led to the original counter example to Piccards theorem. Thus was discovered a new facet of the nature of possible diffraction pattern ambiguities in crystal structures.

Remarks

- The second direction taken in labeling graphs on *n* vertices maintained the requirements of graceful labelings. Instead, it was determined that approximately 1/3 of the edges of K_n needed to be eliminated for the remaining graph to be graceful/ knowledge that 2/3(nC2) is the limit for the number of egdes in a graceful graph, in turn, gives design limits for communication networks of this type.

- The extension of graceful labelings to directed graphs arose in the characterization of some algebraic structures. (Hsu and Keedwell) The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near complete mappings finite Neofields etc., are discussed in Bloom and Hsu.

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- 1. G.S.Bloom ans D.F. Hsu, On graceful digraphs and a problem in network addressing, Congr. Numer., Vol. 35(1982), 91-103.
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Graceful directed graphs

- A directed graph D with n vertices ands e edges, no self-loops and multiple (unless mentioned otherwise) edges is labeled by assigning to each vertex a distinct element from the set $Z_{e+1} = \{0,1,2,\ldots,e\}$. An edge (x, y) from vertex x to y is labeled with $\lambda(xy) = \lambda(x) - \lambda(y) \pmod{(e+1)}$, where $\lambda(x)$ and $\lambda(y)$ are the values assigned to the verticess x and y. A labeling is a graceful labeling of D if all $\lambda(xy)$ are distinct. Then D is called a graceful digraph.

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- In general, labels for edges in undirected graphs are defined by using some symmetric functions labels of the end points, eg., absolute difference or modular sum.
- For labeling a digraph, that symmetry should be removed. Moreover, a labeling of directed graphs analogous to the graceful labeling can be realized by demanding that arc labels be limited in value to the range of the node labels. Both of these features are realized by modular subtraction.

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- There are many ways to get gracefully labeled digraphs, both simple and sophisticated. An example of the former is to start with any gracefully labeled undirected graph *G* with node labeling $\lambda(x)$ for node x. Simply orienting the edges of *G* to point toward the larger node value produces a graceful digraph *D* with *G* as its underlying graph. Thus, if $\lambda(x) > \lambda(y)$, then the edge *xy* is labeled $\lambda(xy) = \lambda(x) - \lambda(y) = |\lambda(x) - \lambda(y)|$ which results in the same value being assigned to the corresponding edges in *G* and *D*.

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- There is another class of digraphs that are immediately gracefully numbered, if one knows a graceful labeling of their underlying graphs. A bidirectional digraph $\underline{G}$ based on (underlying) graph *G* has the same node set as *G*, but has arcs (*x*, *y*) and (*y*, *x*) replacing each edge of *G*. Unlike the previous example, the modularity in subtraction is explicitly used here, and it is easy to show the following.

- If *G* is a gracefully labeled graph, then $\underline{G}$ is graceful with the same node labels.

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- On the other hand it is not always true that the underlying graph of a graceful digraph is graceful. For example, it is known that the cycle with 6 vertices is not graceful but it is digraceful for at least one orientation of its arcs. Moreover, even in those cases that the underlying graph of a digraph is graceful, it is rare that the node labeling of the graceful digraph will also serve as a graceful labeling of the underlying graph. For example consider the graceful labeling and orientation of the cycle with 4 vertices. Even though this is disgraceful, this labeling will not work of the underlying cycle. (but the above cycle is graceful)
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- Some digraphs are graceful even when their underlying graphs are not graceful. For example, complete graphs with fewer than four vertices are graceful, but for greater than four they are not graceful. But there are complete graphs which are digraceful for more than four vertices.

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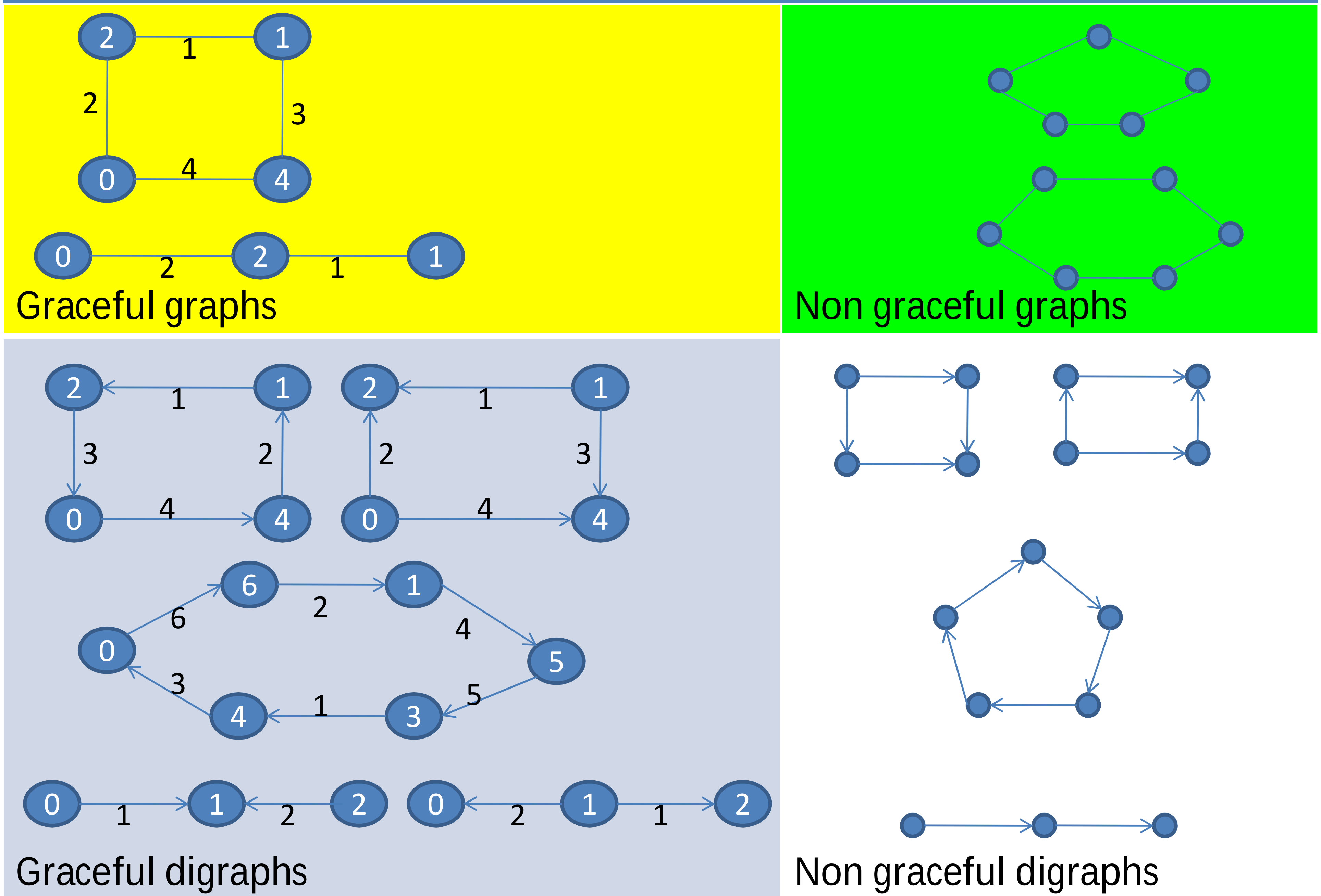
Fig: A Selection of graphs and digraphs classified as graceful or not.



- Trees: The most studied problem for graceful undirected graphs is to determine if all trees are graceful. One can see that if a graceful labeling for a tree is established then it can be made disgraceful with simple orientation. Beyond the fact that graceful trees trivially give graceful directed trees, little is known about general, arbitrarily oriented trees. Even though little specific has been known about the graceful labeling of directed trees, the conjecture that " all trees are digraceful" seems plausible. This is a weaker conjecture than the one claims that " all trees are graceful" If the stronger conjecture holds, then the weaker conjecture is true by using a trivial orientation. But even if the stronger conjecture is false, it is nevertheless possible for nontrivial edge orientation of ungraceful trees to give graceful digraphs.

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- Only one infinite class of graceful directed trees has been characterized. A directed path is **unidirectional** if all internal nodes have **in degree** = **out degree** =1.

- **Theorem**: The unidirectional path P_n on n vertices is graceful if and only if n is even.
- The labels of the consecutive nodes are given by $\lambda(v_i)=(-1)^{n+1}\lfloor i/2 \rfloor \pmod{n}$. Type equation here.

- A non equivalent graceful labeling of a unidirectional path can also be generated by the process of sequencing the elements of a sequenceable cyclic group.

- The procedure for using sequenceable cycles group to generate graceful labelings for the unidirectional path can be viewed as a special class of “ruler models” using the additive group of integers modulo n..

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- Definition: A finite group $(G, *)$ of order n is said to be sequenceable if its elements can be arranged in a sequence $a_0 = e, a_1$, a_2 , ..., a_{n-1} in such a way that the partial products $b_0 = a_0$, $b_1 = a_0 * a_1$,

  $b_2 = a_0 * a_1 * a_2, \ldots, b_{n-1} = a_0 * a_1 * \ldots * a_{n-1}$

are all distinct.

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- The following is an alternative way of stating the above theorem
- Theorem: The unidirectional path is graceful iff Z_n is sequenceable.
- Example: Consider the set $\{a_i\}$ = {0,1,6,3,4,5,2,7} which is a sequencing of the cyclic group Z_8
- Consequently, $\{b_i = \sum a_k (\text{mod}\, n)\}$= {0,1,7,2,6,3,5,4} is used to label the vertices of P_8 .

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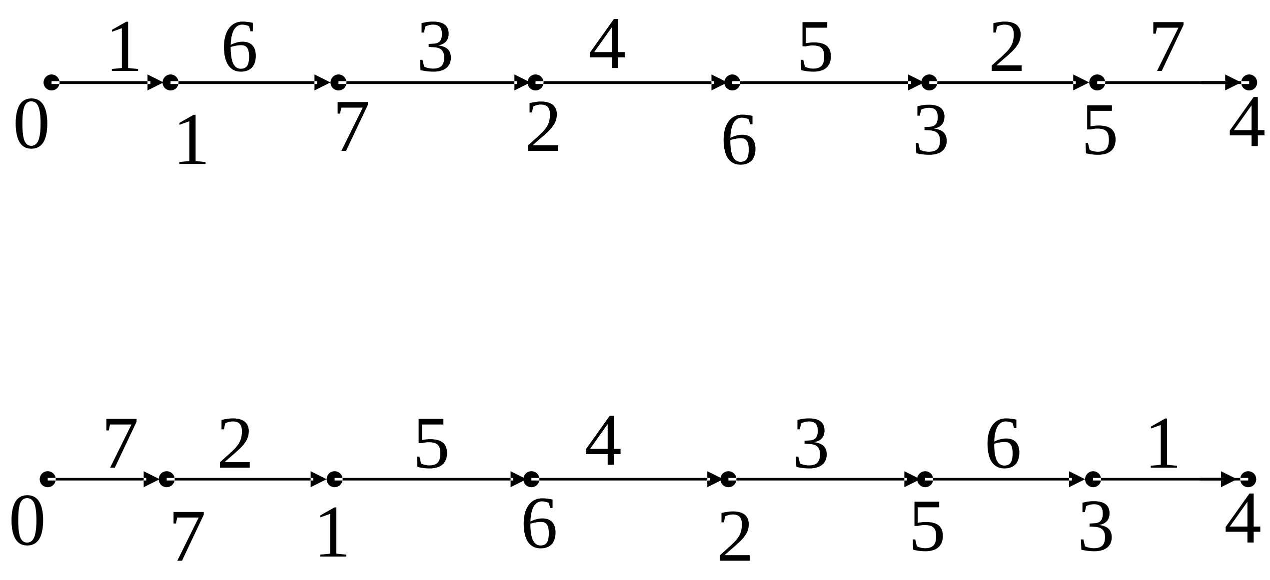
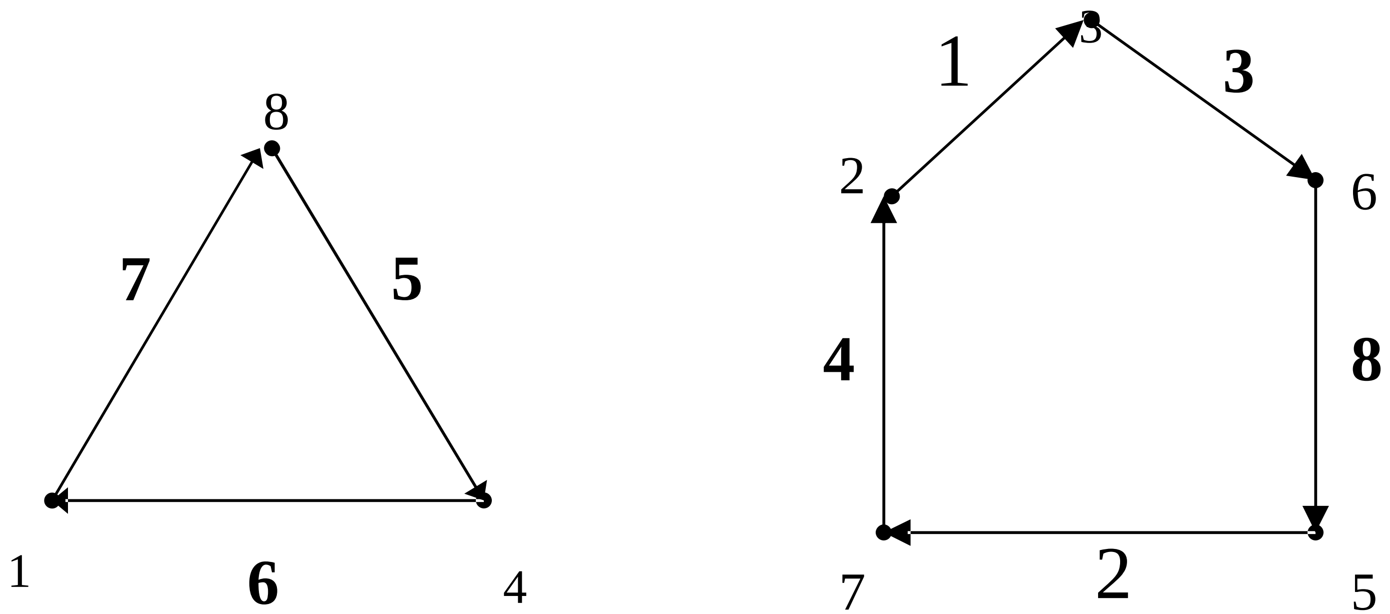

1 6 3 4 5 2 7
0 1 7 2 6 3 5 4
7 2 5 4 3 6 1
0 7 1 6 2 5 3 4

- Union of unicycles: Unidirectional cycles (or unicycles) are connected digraphs in which every vertex has indegree = outdegree =1. Some unicysles are graceful and some are not. Moreover, some collections of disjoint unicyclic components are graceful and some are not.
- Theorem: For a union of unicycles to be graceful, it is necessary that the total number of edges in the digraph be even.

- definition: For a specified integer θ and a sequence $K=\{k_1,k_2,\ldots,k_t\}$ in which k_i are integers such that $\sum k_i = \theta(n-1)$, a (K, θ) complete mapping is an arrangement of θ copies of the nonzero elements of Z_n into t cyclic sequences of lengths $k_1,k_2,\ldots,k_t$, say $(g_{11},g_{12},\ldots,g_{1k1})$ $(g_{21},g_{22},\ldots,g_{2k2}),\ldots,$ $(g_{t1},g_{t2},\ldots g_{tkt})$, such that the following distinct difference property holds. for $i = 1,2,\ldots,t$ and $g_{i,}(k_i+1)=g_i,1$, the set of differences $\{g_{i,j+1}-g_{i,j}\}$ comprises θ copies of the nonzero elements of Z_n.

- We can derive a relation between graceful unicycles and complete mappings by establishing the relation of each to a particular class of permutations.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- For example, if the edge numbers are ignored it can be regarded as the permutation (184)(23657) of $Z_9 \backslash \{0\}$.

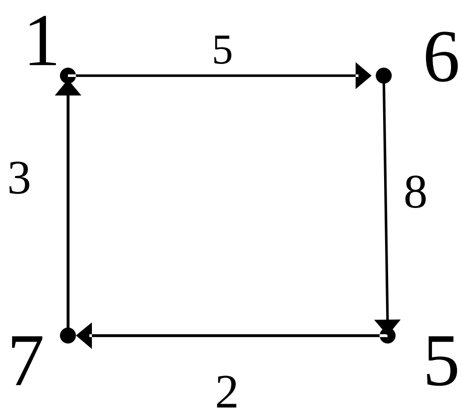
Some results……

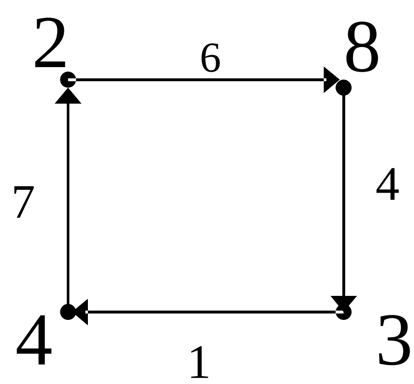
- A graceful labeling for $\cup C_{ki}$, where $\sum k_i = e$, (i=1,2,…,t) exists if and only if there exists a (K,1) complete mapping for Z_{e+1} where $K = \{k_1, k_2, \ldots, k_t\}$.
- A graceful digraph D comprising a collection of both unicycles and unidirectional paths must contain exactly one path and an odd total number of edges.

- In other words, in the special case for $\theta=1$, a $(K,1)$ complete mapping is a permutation of $Z_n \backslash \{0\}$ with t cycles, in which the set of modular differences between successive elements in the cycle equals $Z_n \backslash \{0\}$. (in the above figure it is shown that the example is a permutation which satisfies the distinct difference property). In fact when $\theta=1$, the distinct difference property is equivalent to requiring that all edge numbers be distinct in the graphical representation of the permutation cycles. Consequently, as a direct result of the definition, the following characterization holds:

- Theorem: A graceful labeling for $\cup\, Ck_i$, where $\sum k_i = e$, (i=1,2,…,t) exists if and only if there exists a $(K,1)$ complete mapping for Z_{e+1} where $K = \{k_1, k_2, \ldots, k_t\}$.
- Theorem: Let $G = \cup\, C_i$, (i=1,2,...,t) the union of t disjoint identical unicycles on n vertices. Then G is graceful if (a) t=1 and n is even, (b) t=2, or (c) n=2 or n=6. Moreover, G is not graceful if tn is odd.

- Example: One can see that (1657)(2834) is a $(K,1)$ complete mapping of Z_9 . Where $K=\{4,4\}$. Hence (1657) and (2834) are cyclical vertex sequences that give a graceful labeling of the unidirectional
- $C_4 \cup C_4$.



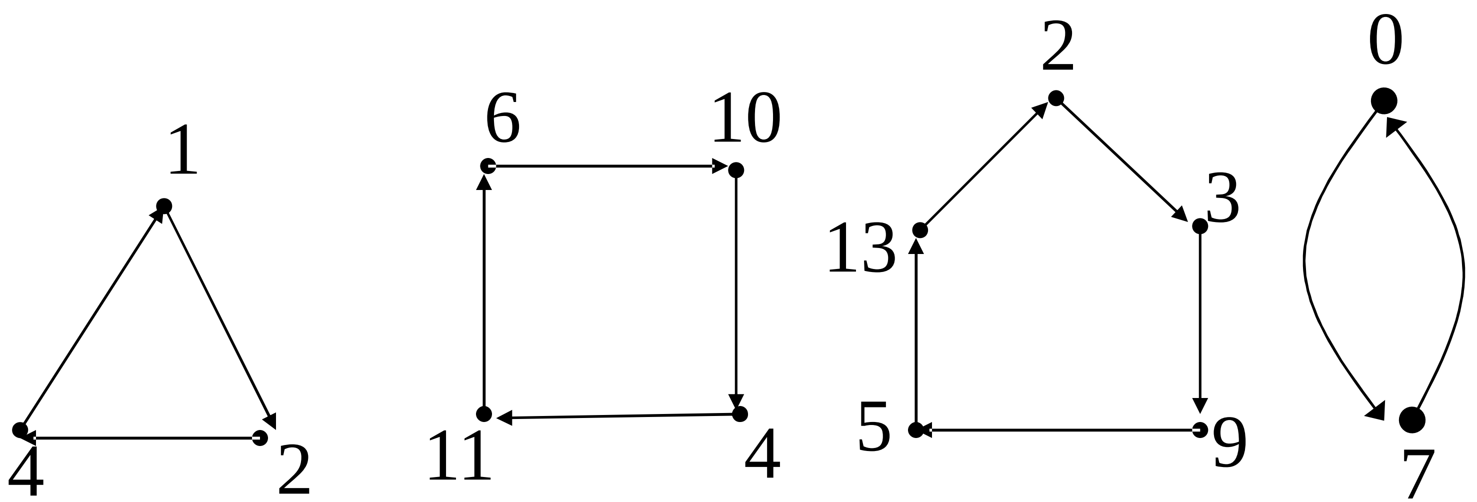


- Collections of unicycles and paths. The graceful labelings of this collection and the $(K,1)$ near complete mappings are related:
- Theorem: Let n and e represents the vertices and edges of a digraph. A graceful labeling of $(\cup Ck_i)\cup(\cup Ph_j)$ $(i=1,2,\ldots,r,\ j=1,2,\ldots,s)$, where $\sum k_i+\sum h_j= n = e+s$, occurs if and only if there exists a $(K,1)$ near complete mapping of $Z_n = Z_{e+s}$, where $K =\{ k_1,k_2,\ldots,k_r;h_1,h_2,\ldots,h_s\}$.

- Theorem: A graceful digraph *D* comprising a collection of both unicycles and unidirectional paths must contain exactly one path and an odd total number of edges.

- A generalized complete mapping is either a $(K,1)$ complete mapping or a $(K,1)$ near complete mapping.

- Example: A (K,1) near mapping of Z_{14} for K={3,4,5,2} is (124)(6 10 8 11)(3 9 5 13 2)[07] which provides a graceful labeling of $C_3 \cup C_4 \cup C_5 \cup P_2$



- **Definition.** *A digraph D with* $|V(D) \cup E(D)| = t$ *is called k-sequential if there exists a bijection* $g: |V(D) \cup E(D)| \rightarrow \{k, k+1, \ldots, t+k-1\}$ *such that each arc* (x, y) *is labeled with*

$$g(x, y) = (g(y) - g(x)) (mod\ t + k).$$

If digraph D admits a k-sequential labeling then D is a k-sequential digraph.

- *A digraph that is* 1-*sequential is called sequential*

Problems

- **Prove or Disprove:** Unidirectional path P_n is sequential.
- Characterize sequential digraphs.
- Count the number of sequential labelings for a given digraph.

Concluding Remarks….

- Graceful digraphs provide a plethora of possibilities for further exploration. For example:

- Graceful digraphs are characterized by a canonical form of their adjacency matrices. Moreover, a subset of these matrices give solutions to a constrained "n – queens" problem.

- Graceful digraphs generated classes of combinatorial designs. There are also possibilities to loosen constraints in investigating graceful digraphs

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Concluding Remarks…

The following questions are currently unanswered:

- How many distinct graceful numberings does a designated graceful digraph have?
- For which classes of undirected graphs can graceful orientations always be found?
- What is the probability that a digraph is graceful?
- What other mathematical and “real world” application can be determined for graceful digraphs?

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