Graph Colorings

Niranjan Balachandran

Indian Institute of Technology Bombay.

What is Graph Coloring?

Suppose G is a graph. Let k be a positive integer. Denote $[k] := \{1, 2, \ldots, k\}$.

What is Graph Coloring?

Suppose G is a graph. Let k be a positive integer. Denote $[k] := \{1, 2, \ldots, k\}$.

Definition

k-**coloring**: *A map* $\phi : V(G) \to [k]$ *such that if* $u \leftrightarrow v$ *in* G *then* $\phi(u) \neq \phi(v)$.

What is Graph Coloring?

Suppose G is a graph. Let k be a positive integer. Denote $[k] := \{1, 2, \ldots, k\}$.

Definition

k-**coloring**: *A map* $\phi : V(G) \to [k]$ *such that if* $u \leftrightarrow v$ *in* G *then* $\phi(u) \neq \phi(v)$.

Definition

Chromatic number of G: *The minimum* k *such that there is a* k*-coloring of* G.

The Chromatic number is denoted by $\chi(G)$.

Example: The Petersen Graph

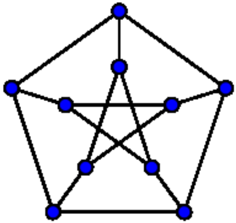


Figure: The Petersen Graph

Example: The Petersen Graph

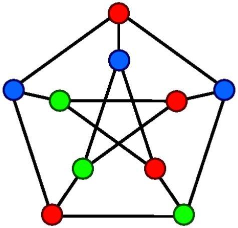


Figure: Petersen Graph with a 3-coloring.

Example: The Petersen Graph

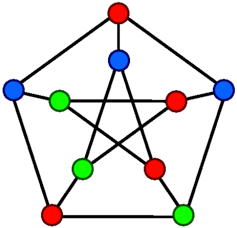


Figure: Petersen Graph with a 3-coloring. $\chi(\text{Petersen}) = 3$.

Simplest cases: Graphs with $\chi(G) = 1$ and $\chi(G) = 2$

- If $\chi(G) = 1$ then G has no edges.

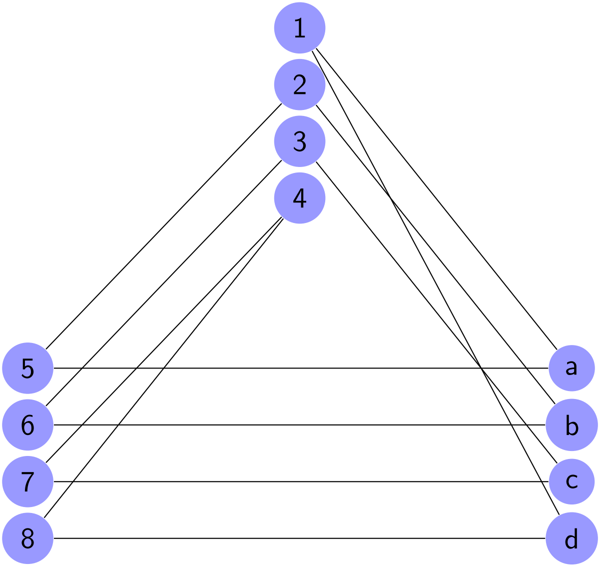
Simplest cases: Graphs with $\chi(G) = 1$ and $\chi(G) = 2$

- If $\chi(G) = 1$ then G has no edges.
- If $\chi(G) = 2$ then G is non-trivial *bipartite*.

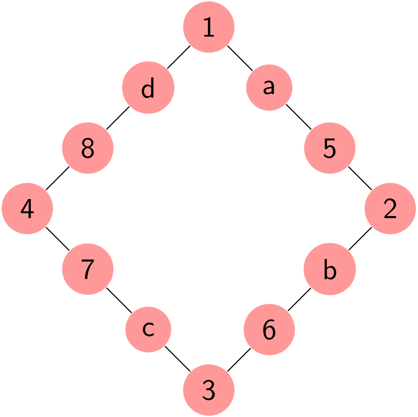
Simplest cases: Graphs with $\chi(G) = 1$ and $\chi(G) = 2$

- If $\chi(G) = 1$ then G has no edges.
- If $\chi(G) = 2$ then G is non-trivial *bipartite*.
- Bad news: No 'nice' characterization for graphs of chromatic number k for any $k \geq 3$.

Why no nice characterization?



Why no nice characterization?



An upper bound from local considerations

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider coloring the vertices one at a time...

An upper bound from local considerations

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider coloring the vertices one at a time...greedily...

An upper bound from local considerations

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider coloring the vertices one at a time...greedily...

- Proposition

  $\chi(G) \leq \Delta + 1$, *where* $\Delta = \max_{v \in V} d(v)$.

An upper bound from local considerations

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider coloring the vertices one at a time...greedily...

- Proposition
  $\chi(G) \leq \Delta + 1$, *where* $\Delta = \max_{v \in V} d(v)$.
- Theorem
  (Brooks): If $G \neq C_{2n+1}, K_n$ *and is connected then* $\chi(G) \leq \Delta$.

Lower bounds

- If $H \subset G$ then $\chi(G) \geq \chi(H)$. In particular, $\chi(G) \geq \omega(G)$ where $\omega(G)$ is the size of a maximum clique in G.

Lower bounds

- If $H \subset G$ then $\chi(G) \geq \chi(H)$. In particular, $\chi(G) \geq \omega(G)$ where $\omega(G)$ is the size of a maximum clique in G.
- $\chi(G) \geq \frac{n}{\alpha(G)}$, where $\alpha(G) =$ Size of a maximum independent set in G.

Question: Does there exist a graph G with no triangles (no K_3 as a subgraph) and with chromatic number, say 1000?

Question: Does there exist a graph G with no triangles (no K_3 as a subgraph) and with chromatic number, say 1000?

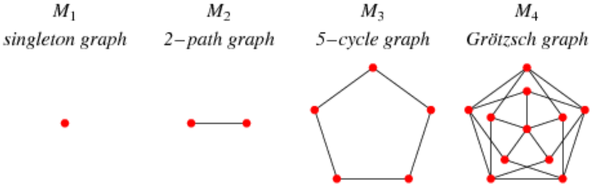


Figure: The Mycielski construction for $\chi(G) = 1, 2, 3, 4$.

Graphs with no small cycles and large chromatic number

Theorem

(Blanche Descartés akaTutte) There exists graphs with girth 6 and chromatic number k for any $k \geq 2$.

Graphs with no small cycles and large chromatic number

Theorem

(Blanche Descartés akaTutte) There exists graphs with girth 6 and chromatic number k for any $k \geq 2$.

Theorem

(Erdős) For any given k, g there exists a graph G with girth greater than g and $\chi(G) \geq k$.

Sketch of proof of Erdős' result

- Pick G *randomly*, i.e., pick each edge independently, and with probability p.

Sketch of proof of Erdős' result

- Pick G *randomly*, i.e., pick each edge independently, and with probability p.
- If $N =$ number of cycles of size less than or equal to g, then $\mathbb{E}(N) = \sum_{i=3}^{g} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i < \frac{gn^{g\theta}}{6}$ if we have $p = n^{\theta-1}$ (for some $0 < \theta < 1$).

Sketch of proof of Erdős' result

- Pick G *randomly*, i.e., pick each edge independently, and with probability p.
- If $N =$ number of cycles of size less than or equal to g, then $\mathbb{E}(N) = \sum_{i=3}^{g} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i < \frac{gn^{g\theta}}{6}$ if we have $p = n^{\theta-1}$ (for some $0 < \theta < 1$).
- In particular, if $\theta < 1/g$ we have $\mathbb{E}(N) = o(n)$, so $\mathbb{P}(N > n/2) < 0.1$, say.

Sketch of proof of Erdős' result (contd.)

- $$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left(ne^{-(p(x-1)/2}\right)^{x} < 0.1,$$

  say, if $x = Cn^{1-\theta}\log n$ for a suitable constant C.

Sketch of proof of Erdős' result (contd.)

- $$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left(ne^{-(p(x-1)/2}\right)^x < 0.1,$$

  say, if $x = Cn^{1-\theta}\log n$ for a suitable constant C.

- Delete from each small cycle an edge to destroy all cycles of size at most g (deleting at most $n/2$ vertices). The resulting graph G^* has $\alpha(G^*) < Cn^{1-\theta}\log n$ and has no cycles of size less than or equal to g. Furthermore, $\chi(G) \geq \chi(G^*) \geq \frac{n/2}{Cn^{1-\theta}\log n}$. ■

Lovász gave a (complicated) constructive proof.

Lovász gave a (complicated) constructive proof.

Nešetřil and Rödl gave a simpler constructive proof.

Lovász gave a (complicated) constructive proof.

Nešetřil and Rödl gave a simpler constructive proof.

No known ‘purely graph-theoretic’ constructions.

Lovász gave a (complicated) constructive proof.

Nešetřil and Rödl gave a simpler constructive proof.

No known 'purely graph-theoretic' constructions.

The Erdős result actually proves that almost all graphs are very 'close' to such graphs!

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

YES, IT CAN BE MUCH, MUCH LARGER!

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

YES, IT CAN BE MUCH, MUCH LARGER!

Theorem

(Erdős) Given any $k \geq 3$ there exists $\epsilon = \epsilon(k) > 0$ and $n_0 = n_0(\epsilon)$ such that the following holds: For every $n \geq n_0$ there exists a graph G on n vertices satisfying

1. *For every subset H of at most ϵn vertices $\chi(H) \leq 3$.*

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

YES, IT CAN BE MUCH, MUCH LARGER!

Theorem

(Erdős) Given any $k \geq 3$ there exists $\epsilon = \epsilon(k) > 0$ and $n_0 = n_0(\epsilon)$ such that the following holds: For every $n \geq n_0$ there exists a graph G on n vertices satisfying

1. *For every subset H of at most ϵn vertices $\chi(H) \leq 3$.*
2. $\chi(G) \geq k(!)$.

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

YES, IT CAN BE MUCH, MUCH LARGER!

Theorem

(Erdős) Given any $k \geq 3$ there exists $\epsilon = \epsilon(k) > 0$ and $n_0 = n_0(\epsilon)$ such that the following holds: For every $n \geq n_0$ there exists a graph G on n vertices satisfying

1. *For every subset H of at most ϵn vertices $\chi(H) \leq 3$.*
2. $\chi(G) \geq k(!)$.

- Proof uses a probabilistic construction.

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

YES, IT CAN BE MUCH, MUCH LARGER!

Theorem

(Erdős) Given any $k \geq 3$ there exists $\epsilon = \epsilon(k) > 0$ and $n_0 = n_0(\epsilon)$ such that the following holds: For every $n \geq n_0$ there exists a graph G on n vertices satisfying

1. *For every subset H of at most ϵn vertices $\chi(H) \leq 3$.*
2. $\chi(G) \geq k(!)$.

- Proof uses a probabilistic construction.
- Almost every graph (in the random graph model) can be altered mildly to obtain such a G.

Any improvements on Brooks' theorem?

Any improvements on Brooks' theorem?

Characterizing graphs for which $\chi(G) \leq \Delta - 1$ is difficult.

Any improvements on Brooks' theorem?

Characterizing graphs for which $\chi(G) \leq \Delta - 1$ is difficult.

Theorem

(Maffray, Preissmann) Determining if a 4-regular graph has chromatic number 3 is NP-complete.

Any improvements on Brooks' theorem?

Any improvements on Brooks' theorem?

Theorem

(Molly,Reed) For $\Delta \gg 0$ *and* $\omega(G) \leq \Delta - 1$ *we have* $\chi(G) \leq \Delta - 1$.

Any improvements on Brooks' theorem?

Theorem

(Molly,Reed) For $\Delta \gg 0$ *and* $\omega(G) \leq \Delta - 1$ *we have* $\chi(G) \leq \Delta - 1$.

Theorem

(J.H. Kim) If G *has girth at least* 5*, then* $\chi(G) \leq \frac{\Delta}{\log \Delta}(1 + o(1))$ *for* $\Delta \gg 0$.

Any improvements on Brooks' theorem?

Theorem

(Molly,Reed) For $\Delta \gg 0$ and $\omega(G) \leq \Delta - 1$ we have $\chi(G) \leq \Delta - 1$.

Theorem

(J.H. Kim) If G has girth at least 5, then $\chi(G) \leq \frac{\Delta}{\log \Delta}(1 + o(1))$ for $\Delta \gg 0$.

Theorem

(Johansson) If G is triangle free, then $\chi(G) \leq O\left(\frac{\Delta}{\log \Delta}\right)$.

Any improvements on Brooks' theorem?

Theorem

(Molly,Reed) For $\Delta \gg 0$ *and* $\omega(G) \leq \Delta - 1$ *we have* $\chi(G) \leq \Delta - 1$.

Theorem

(J.H. Kim) If G *has girth at least* 5, *then* $\chi(G) \leq \frac{\Delta}{\log \Delta}(1 + o(1))$ *for* $\Delta \gg 0$.

Theorem

(Johansson) If G *is triangle free, then* $\chi(G) \leq O\left(\frac{\Delta}{\log \Delta}\right)$.

All these proofs heavily rely on probabilistic techniques.

Proof of Kim: An Iterative coloring process

- Pick a small subset of uncolored vertices (i.e. pick each uncolored vertex with probability $\frac{O(1)}{\log \Delta}$) and for each of these chosen vertices, assign a color chosen uniformly at random from the list of colors available for that vertex. Initially each $|L_v| = \frac{\Delta}{\log \Delta}(1+\epsilon)$ for each v.

Proof of Kim: An Iterative coloring process

- Pick a small subset of uncolored vertices (i.e. pick each uncolored vertex with probability $\frac{O(1)}{\log \Delta}$) and for each of these chosen vertices, assign a color chosen uniformly at random from the list of colors available for that vertex. Initially each $|L_v| = \frac{\Delta}{\log \Delta}(1+\epsilon)$ for each v.
- 'Uncolor' any vertex if one of its neighbors was also picked and assigned the same color (this will hold for both of these vertices). Remove that color from the list of colors that vertex may be assigned in future (again, for both the vertices).

Proof of Kim: An Iterative coloring process

- If a vertex v is assigned a color and retains it (after step 2), remove this color from the assignable list of colors of any of its remaining neighbors.

Proof of Kim: An Iterative coloring process

- If a vertex v is assigned a color and retains it (after step 2), remove this color from the assignable list of colors of any of its remaining neighbors.
- With positive probability, this iteration can be carried about $O\left(\frac{\log \Delta}{\log\log \Delta}\right)$ times ensuring that after each iteration $|L_v|$ roughly the same for each uncolored v, and the number of uncolored neighbors of v which share some color c in the list L_v is 'much smaller' $|L_v|$.

Proof of Kim: An Iterative coloring process

- If a vertex v is assigned a color and retains it (after step 2), remove this color from the assignable list of colors of any of its remaining neighbors.
- With positive probability, this iteration can be carried about $O\left(\frac{\log \Delta}{\log \log \Delta}\right)$ times ensuring that after each iteration $|L_v|$ roughly the same for each uncolored v, and the number of uncolored neighbors of v which share some color c in the list L_v is 'much smaller' $|L_v|$.
- The remaining final piece of the graph can be colored greedily.

Some Open Problems

Some Open Problems

1. (Hadwiger's conjecture) Let $\mathcal{G}$ be a class of graphs closed under deletions of vertices/edges, and contractions of edges, and removing any loops that might arise. Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices in a largest clique that occurs in $\mathcal{G}$.

Some Open Problems

1. (Hadwiger's conjecture) Let $\mathcal{G}$ be a class of graphs closed under deletions of vertices/edges, and contractions of edges, and removing any loops that might arise. Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices in a largest clique that occurs in $\mathcal{G}$.
2. (B. Reed) $\chi(G) \leq \left\lceil \frac{\Delta+\omega+1}{2} \right\rceil$, where $\omega = \omega(G)$ is the size of a maximum clique in G.

Some Open Problems

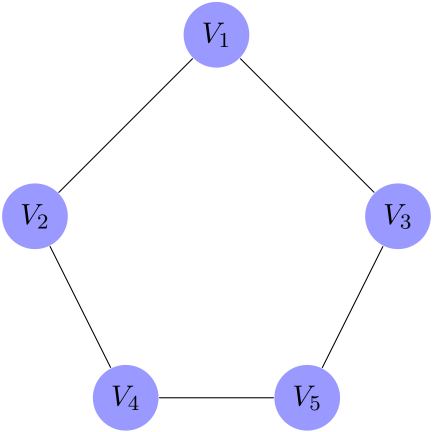
1. (Hadwiger's conjecture) Let $\mathcal{G}$ be a class of graphs closed under deletions of vertices/edges, and contractions of edges, and removing any loops that might arise. Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices in a largest clique that occurs in $\mathcal{G}$.
2. (B. Reed) $\chi(G) \leq \lceil \frac{\Delta+\omega+1}{2} \rceil$, where $\omega = \omega(G)$ is the size of a maximum clique in G.
3. (Borodin, Kostochka) If $\Delta \geq 9$ then $\chi(G) \leq \Delta - 1$. The Reed-Molloy result proves this asymptotically. Bounds in that proof are too large.

Some Open Problems

1. (Hadwiger's conjecture) Let $\mathcal{G}$ be a class of graphs closed under deletions of vertices/edges, and contractions of edges, and removing any loops that might arise. Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices in a largest clique that occurs in $\mathcal{G}$.
2. (B. Reed) $\chi(G) \leq \lceil \frac{\Delta+\omega+1}{2} \rceil$, where $\omega = \omega(G)$ is the size of a maximum clique in G.
3. (Borodin, Kostochka) If $\Delta \geq 9$ then $\chi(G) \leq \Delta - 1$. The Reed-Molloy result proves this asymptotically. Bounds in that proof are too large.
4. Any better lower bounds on $\chi(G)$?

Any improvements on Brooks' theorem?

9 is best possible in the Borodin-Kostochka conjecture:



List Colorings of Graphs

Let $\mathcal{C}$ be a set of colors, and for each $v \in V(G)$, let $L_v \subset \mathcal{C}$. Let $\mathcal{L} := \{L_v | v \in V\}$.

List Colorings of Graphs

Let $\mathcal{C}$ be a set of colors, and for each $v \in V(G)$, let $L_v \subset \mathcal{C}$. Let $\mathcal{L} := \{L_v | v \in V\}$.

A *List coloring* ϕ for G is a proper coloring of G with the constraint that $\phi(v) \in L_v$ for each $v \in V$.

Let $\mathcal{C}$ be a set of colors, and for each $v \in V(G)$, let $L_v \subset \mathcal{C}$. Let $\mathcal{L} := \{L_v | v \in V\}$.

A *List coloring* ϕ for G is a proper coloring of G with the constraint that $\phi(v) \in L_v$ for each $v \in V$.

Definition

The **List Chromatic number** *of G (denoted $\chi_l(G)$) :=$\min_k$ G has a list coloring for any collection $\mathcal{L} := \{L_v | v \in V\}$ provided $|L_v| \geq k$, irrespective of the actual lists themselves.*

List Colorings of Graphs

Let $\mathcal{C}$ be a set of colors, and for each $v \in V(G)$, let $L_v \subset \mathcal{C}$. Let $\mathcal{L} := \{L_v | v \in V\}$.

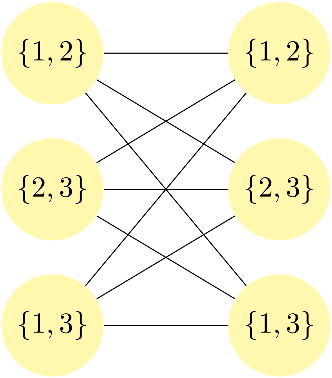
A *List coloring* ϕ for G is a proper coloring of G with the constraint that $\phi(v) \in L_v$ for each $v \in V$.

Definition

The **List Chromatic number** *of G (denoted $\chi_l(G)$) :=$\min_k$ G has a list coloring for any collection $\mathcal{L} := \{L_v | v \in V\}$ provided $|L_v| \geq k$, irrespective of the actual lists themselves.*

If all the lists are identical, then the minimum number k is the definition above is simply $\chi(G)$.

The list chromatic number of a graph can be larger than the chromatic number.



Results on List Colorings

Theorem

(Erdős, Rubin, Taylor): $\chi_l(K_{m,m}) > k$ *if* $m = \Omega(k^2 2^k)$.

Results on List Colorings

Theorem

(Erdős, Rubin, Taylor): $\chi_l(K_{m,m}) > k$ *if* $m = \Omega(k^2 2^k)$.

Erdős, Rubin, and Taylor characterized all the graphs of list chromatic number 2.

Results on List Colorings

Theorem

(Erdős, Rubin, Taylor): $\chi_l(K_{m,m}) > k$ *if* $m = \Omega(k^2 2^k)$.

Erdős, Rubin, and Taylor characterized all the graphs of list chromatic number 2.

Theorem

(Analogue of Brooks' theorem): $\chi_l(G) \leq \Delta$ *if* $G \neq C_{2n+1}, K_n$.

Results on List Colorings

Theorem

(Erdős, Rubin, Taylor): $\chi_l(K_{m,m}) > k$ if $m = \Omega(k^2 2^k)$.

Erdős, Rubin, and Taylor characterized all the graphs of list chromatic number 2.

Theorem

(Analogue of Brooks' theorem): $\chi_l(G) \leq \Delta$ if $G \neq C_{2n+1}, K_n$.

Theorem

(Johansson,Kim): For $\Delta \gg 0$, $\chi_l(G) \leq O\left(\frac{\Delta}{\log \Delta}\right)$ if G is triangle free (resp. girth at least 5).

List Colorings for Bipartite graphs

Proposition

If G is bipartite then $\chi_l(G) \leq \lceil \log_2 |V| \rceil$.

List Colorings for Bipartite graphs

Proposition

If G is bipartite then $\chi_l(G) \leq \lceil \log_2 |V| \rceil$.

A crucial difference between $\chi(G)$ and $\chi_l(G)$: There exist graphs with chromatic number 2 and with minimum degree k for any given k.

Proposition

If G is bipartite then $\chi_l(G) \leq \lceil \log_2 |V| \rceil$.

A crucial difference between $\chi(G)$ and $\chi_l(G)$: There exist graphs with chromatic number 2 and with minimum degree k for any given k.

Theorem

(Alon): If the minimum degree of G is d, then $\chi_l(G) \geq (\frac{1}{2} - o(1)) \log d$.

Some Open problems

1. (Alon) For any bipartite graph G $\chi_l(G) \leq O(\log \Delta)$. The same is known to hold for a random bipartite graph whp.

Some Open problems

1. (Alon) For any bipartite graph G $\chi_l(G) \leq O(\log \Delta)$. The same is known to hold for a random bipartite graph whp.
2. (Ohba) If $|V(G)| \leq 2\chi(G) + 1$ then $\chi_l(G) = \chi(G)$. Erdős, Rubin, taylor proved this for the graph $K_n(2)$; Molloy-Sudakov proved this asymptotically, i.e .if $|V(G)| \leq (2 - o(1))\chi(G)$ then $\chi_l(G) = \chi(G)$.

Some Open problems

1. (Alon) For any bipartite graph G $\chi_l(G) \leq O(\log \Delta)$. The same is known to hold for a random bipartite graph whp.
2. (Ohba) If $|V(G)| \leq 2\chi(G) + 1$ then $\chi_l(G) = \chi(G)$. Erdős, Rubin, taylor proved this for the graph $K_n(2)$; Molloy-Sudakov proved this asymptotically, i.e .if $|V(G)| \leq (2 - o(1))\chi(G)$ then $\chi_l(G) = \chi(G)$.
3. (folklore) What is the list chromatic number of the n-dimensional cube for $n > 3$?

Some Open problems

1. (Alon) For any bipartite graph G $\chi_l(G) \leq O(\log \Delta)$. The same is known to hold for a random bipartite graph whp.
2. (Ohba) If $|V(G)| \leq 2\chi(G) + 1$ then $\chi_l(G) = \chi(G)$. Erdős, Rubin, taylor proved this for the graph $K_n(2)$; Molloy-Sudakov proved this asymptotically, i.e .if $|V(G)| \leq (2 - o(1))\chi(G)$ then $\chi_l(G) = \chi(G)$.
3. (folklore) What is the list chromatic number of the n-dimensional cube for $n > 3$?
4. If G is a bipartite graph and $\mathcal{M}$ is a matching between the two parts of G, $\chi_l(G \cup \mathcal{M}) \leq \chi_l(G) + 1$.