Geometric Graphs

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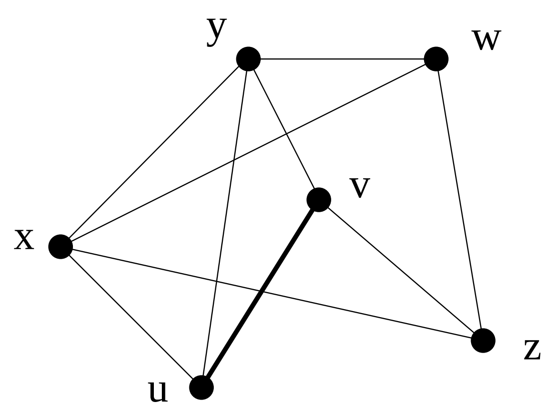
Workshop on Introduction to Graph and Geometric Algorithms
DAIICT, Gandhinagar

Geometric Graph



- ⋆ V = set of geometric objects (point set in the plane)
- ⋆ $E = \{(u, v)\}$ based on some geometric condition

Questions on Geometric Graphs

- ☆ Problems on graphs
  - Independent set, coloring, clique, etc.

- ☆ Combinatorial/Structural questions
  - Obtain Bounds
  - Characterization

- ☆ Computational questions
  - Efficient Algorithm
  - Approximation

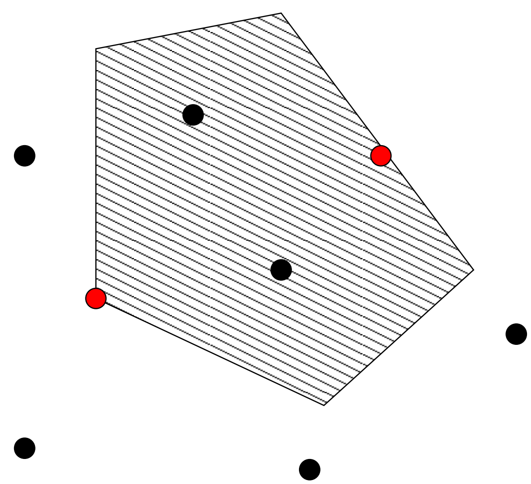
Geometric graphs

- V - set of geometric objects
- E - object i and j satisfy certain geometric condition

- Broad classes of geometric graphs (based on edge condition)
  - Proximity graphs
  - Intersection graphs
  - Distance based graphs

Proximity Graphs

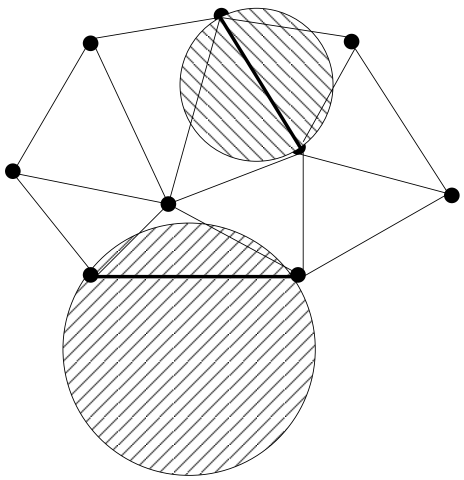
- ☆ P - point set in plane
- ☆ $R_{i,j}$ - proximity region defined by i and j



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $R_{i,j}$ is empty
- ☆ Examples - Delaunay, Gabriel, Relative Neighborhood Graph
- ☆ Applications - Graphics, wireless networks, GIS, computer vision, etc.

Delaunay Graph - Classic Example

- ☆ P - point set in plane



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $\exists$ some empty circle thro' i and j
- ☆ Triangle (i, j, k) if circumcircle(i, j, k) is empty (Equivalent condition)
- ☆ Applications - Graphics, mesh generation, computer vision, etc.

Questions on Delaunay Graph

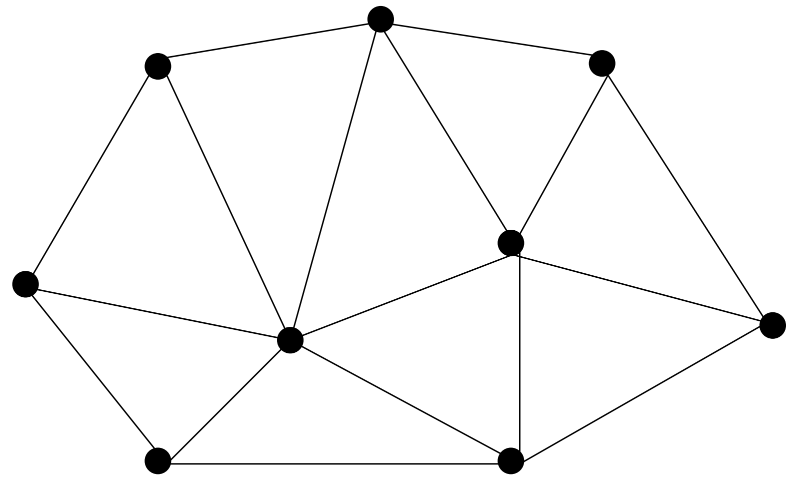
- Combinatorial - Bounds on
  - Maximum size of edge set?
  - Chromatic number?
  - Maximum independent set?

  (Over all possible point sets P)

- Computational
  - Efficient Algorithm

Delaunay Graph - Classic Example

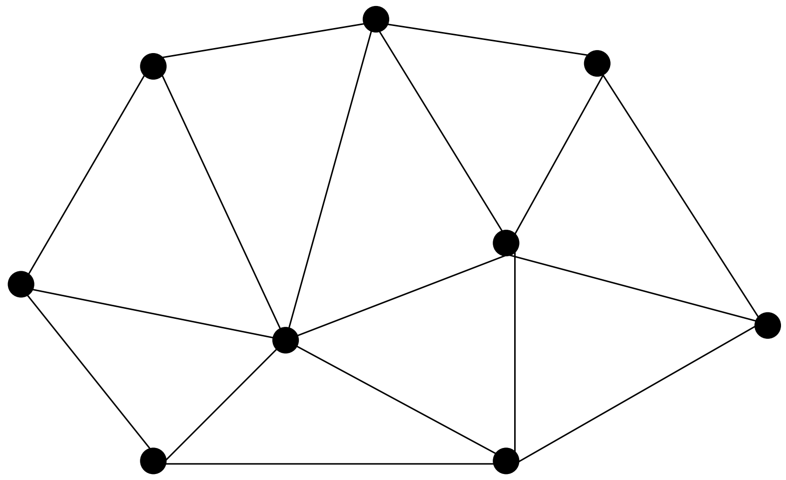
⋆ P - point set in plane



⋆ Observations:

Delaunay Graph - Classic Example

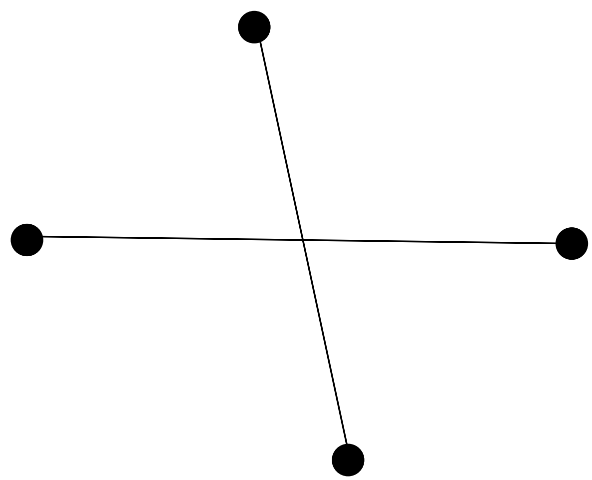
★ P - point set in plane



★ Observations: Planar?

Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



Delaunay Graph - Planar

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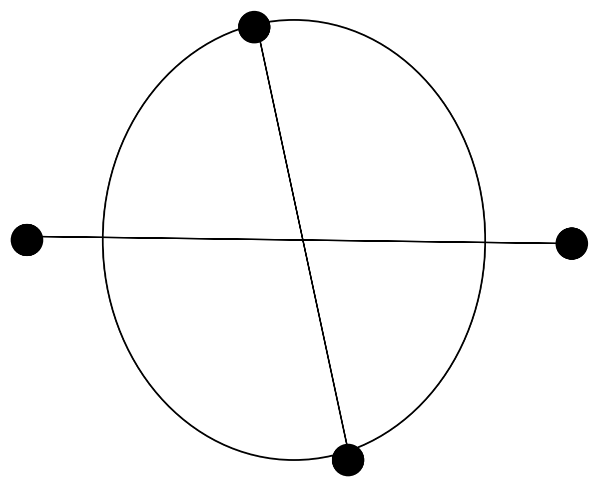

Delaunay Graph - Planar

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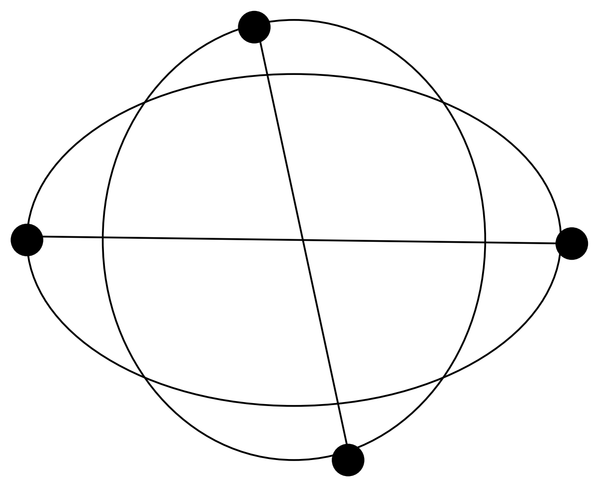
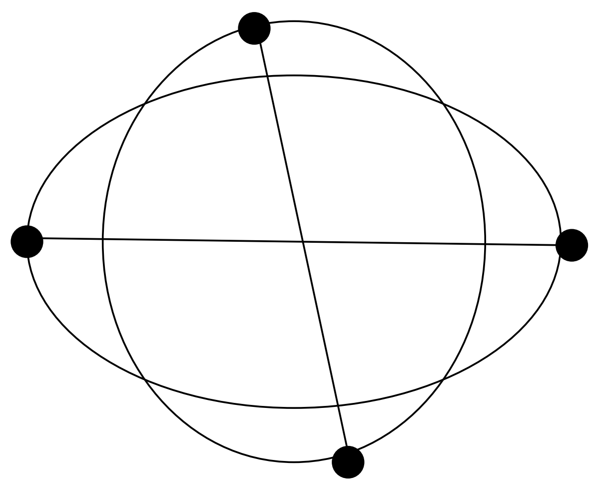


Delaunay Graph - Planar

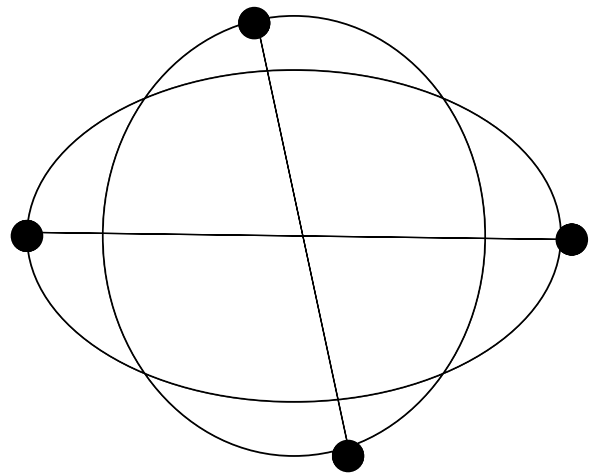
⋆ Let, if possible, 2 edges cross



⋆ Circles c'ant intersect like this (why?)

Delaunay Graph - Planar

- ☆ Let, if possible, 2 edges cross



- ☆ Circles c'ant intersect like this (why?)
- ☆ One endpoint of an edge lies within the other circle
  - Contradiction
- ☆ Alternate proof using angles

Questions on Delaunay Graph

- Given any n-point set P in the plane
  - Delaunay graph is planar

- Maximum size of edge set
  - $\leq 3n - 6$ (Euler's formula)

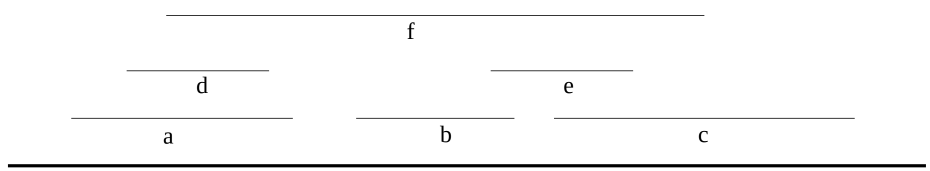
- Chromatic number
  - ≤ 4 (Four color theorem)

- Maximum independent set
  - $\geq n/4$ (Chromatic number)

Intersection Graphs

- ☆ Interval Graph - Classic example

- ☆ S - set of geometric objects s_i (intervals on the real line)

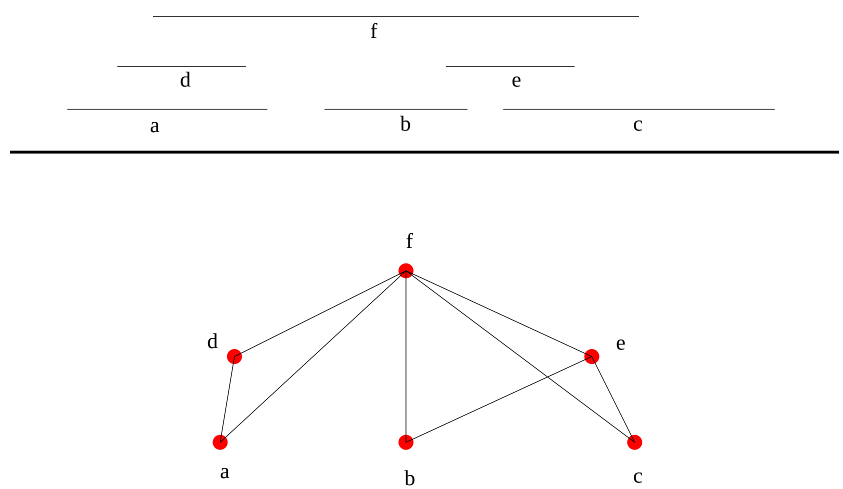


- ☆ V - set of object s_i

- ☆ $(s_i, s_j) \in E$ if objects s_i and s_j intersect

Interval Graphs

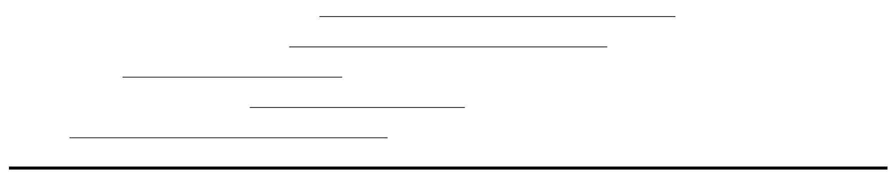
- S - set of intervals on the line



- V - set of object s_i
- $(s_i, s_j) \in E$ if objects s_i and s_j intersect
- Graph problems - Maximum independent set, Maximum clique, Chromatic number, etc.
  - Can be computed efficiently

Intervals

- ⋆ S - set of intervals on the real line
- ⋆ Every 2 intervals in S intersect



- ⋆ Claim: All the intervals have a common intersection

Intervals

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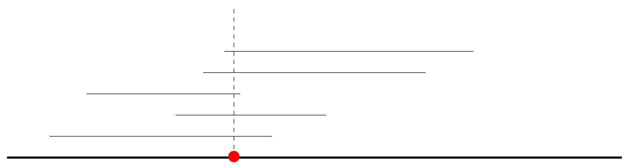

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Intervals

- S - set of intervals on the real line
- Every 2 intervals in S intersect
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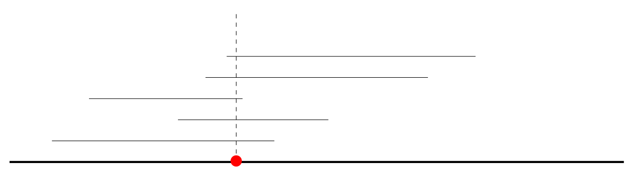
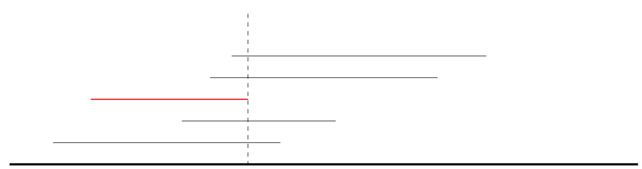


- Induction proof (Exercise)
- Constructive proof
  - Construct a point p that is contained in all the intervals

Intervals

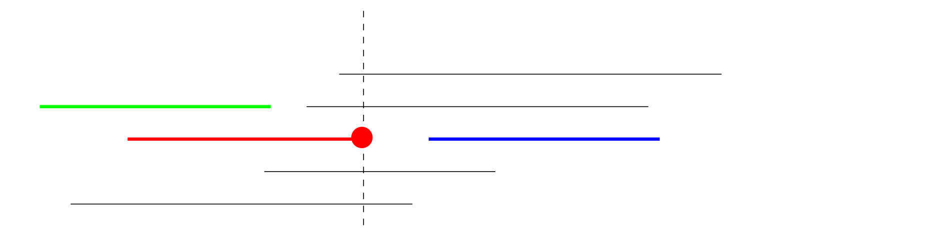
- S - set of intervals on the real line
- Every 2 intervals intersect
- Constructive proof
  - Construct a point p that is contained in all the intervals
- p : Right endpoint of interval that ends first from left
  - Leftmost right endpoint



- Claim: All the intervals contain p

Intervals

- ☆ Construct a point p that is contained in all the intervals
- ☆ p : Right endpoint of interval that ends leftmost
  - Leftmost right endpoint
- ☆ Claim: All the intervals contain p
- ☆ Proof by contradiction



Intersection Graphs of Axis Parallel Rectangles

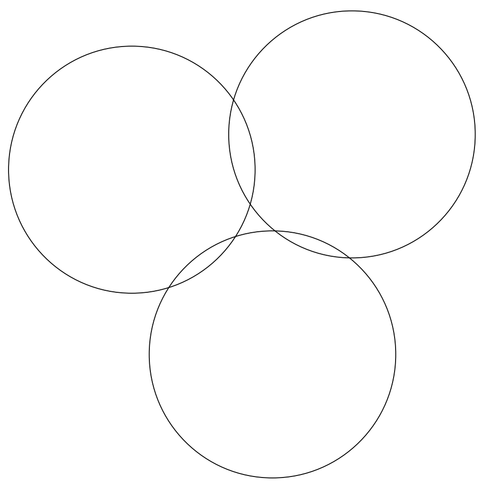
- S - set of axis parallel rectangles
- Every 2 rectangles intersect
  - Claim: There exists a point p contained in all the rectangles
  - Is it true?

Intersection Graphs of Circles

- S - set of circles
- Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles

Intersection Graphs of Circles

- S - set of circles
- Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles
  - Not true



Intersection Graphs of Circles

- S - set of circles
- Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles
  - Not true
- Every 3 circles intersect
  - Claim: There exists a point p contained in all the circles
  - True
- Helly Theorem: Statement true for convex objects

Helly's Theorem

- **Helly's Theorem:** Let C be a collection of convex objects in R^d. If every $d+1$ objects in C have a common intersection, then all the objects in C have a common intersection

- $d = 1$: Intervals in 1D

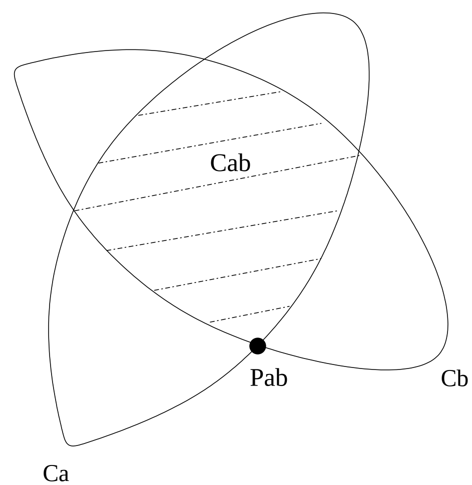
- We will prove for $d = 2$
  - Proof generalizes to d dimensions.

- Induction proof

- Constructive proof
  - Construct a point p that is contained in all the objects

Helly's Theorem in R^2

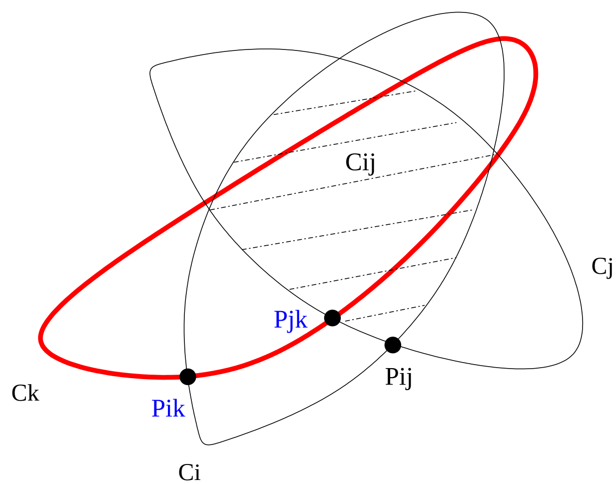
- **Helly's Theorem:** Let C be a collection of convex objects in R^2. If every 3 objects in C have a common intersection, then all the objects in C have a common intersection



- p_{ab} : Lowest point in $C_{ab} = C_a \cap C_b$
- Choose the pair of objects (C_i, C_j) such that p_{ij} is highest among all pairs
- Claim: p_{ij} is contained in all objects in C

Helly's Theorem

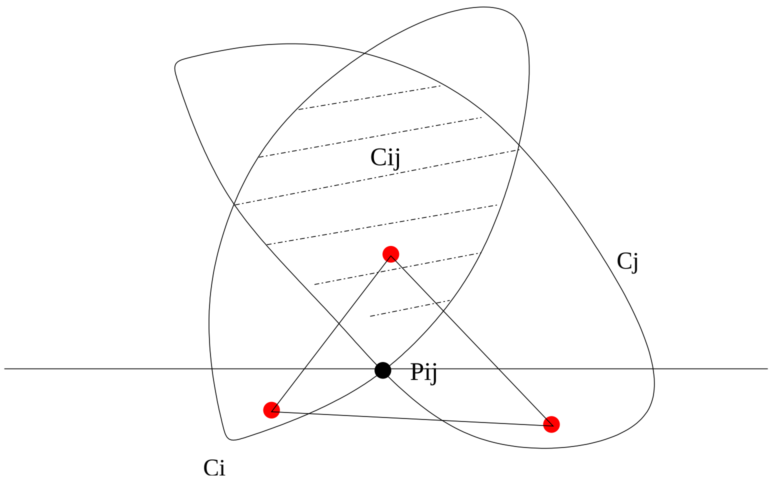
- Claim: p_{ij} is contained in C_k for all k
- $C_{ij} \cap C_k \neq \emptyset$ (Every 3 objects intersect)



- If p_{ij} is not contained in C_k
  - p_{jk} higher than p_{ij} - Contradiction

Helly's Theorem

- Claim: p_{ij} is contained in C_k for all k



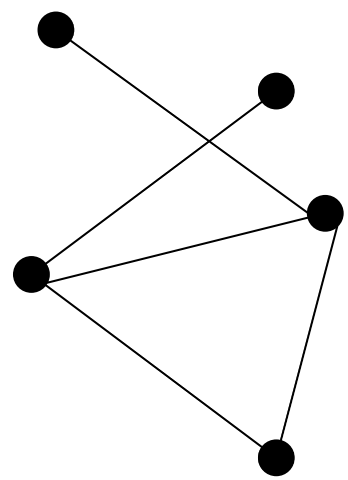
- $C_{ij} \cap C_k \neq \emptyset$ (Every 3 objects intersect)
- C_k intersect both C_i and C_j below p_{ij}
  - p_{ik} and p_{jk} must be lower than p_{ij}
- By convexity, p_{ij} is contained in C_k

Centerpoint Theorem

- ☆ **Centerpoint Theorem:** Let P be a set of n points in the plane. There exists a point p in the plane that is contained in every convex object containing $> \frac{2}{3}n$ points of P
- ☆ Proof:
- ☆ Take any 3 convex objects C_i, C_j, C_k containing $> \frac{2}{3}n$ points
- ☆ $C_i \cap C_j \cap C_k \neq \emptyset$ (Counting argument)
- ☆ Applying Helly theorem, there exists a point p contained in all such convex objects
- ☆ The constant $\frac{2}{3}$ is the best possible

Distance based Graphs

- Unit distance graph
  - V - point set in plane
  - $(i, j) \in E$ if $d(i, j) = 1$

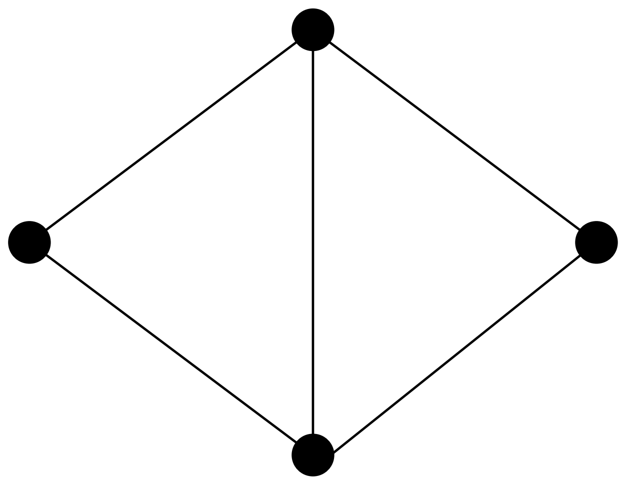


- Place points so as to maximize the number of edges

- Can you get a complete graph? (even for $n = 4$)

Distance based Graphs

- Unit distance graph
  - V - point set in plane
  - $(i, j) \in E$ if $d(i, j) = 1$

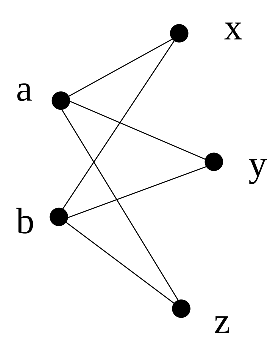


Unit Distance Graph

- V - point set P
- $(i, j) \in E$ if $d(i, j) = 1$

- Maximum number of edges? (Erdos)
  - Over all possible n-point set

- $O(n^{3/2})$ edges
  - Forbidden $K_{2,3}$



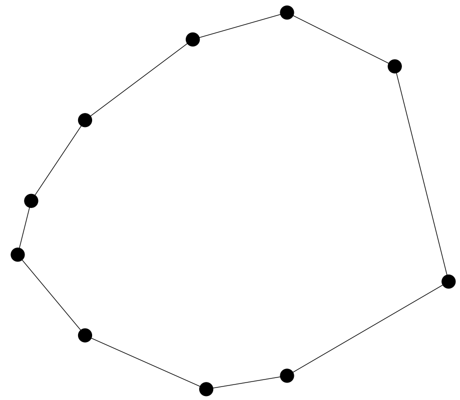
- $O(n^{4/3})$ edges
  - Crossing Lemma, Cuttings, Arrangement of Circles

Unit Distance Graph - Open Problem

- Upper bound
  - $O(n^{4/3})$ edges
- Lower bound
  - $\Omega(n^{1+\frac{c}{\log\log n}})$ [Erdos]
- Conjecture: Lower bound is tight

Unit Distance Graph - Convex Point Set

☆ Convex Point Set



☆ Upper bound: $O(n \log n)$ edges

☆ Lower bound: $2n - 7$ edges

☆ Conjecture: Lower bound is tight ($2n$ edges)

Questions

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