Recognizing and characterizing visibility graphs of simple polygons

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Overview

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2. Computing the configuration space
3. Computing visibility graphs
4. Visibility graph recognition problem
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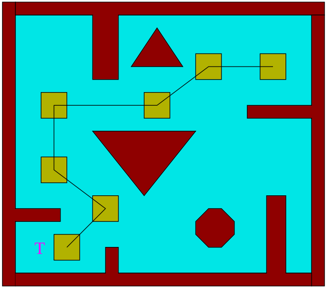
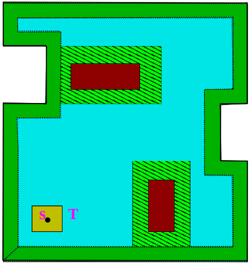
Robot path planning

One of the main problems in robotics, called robot path planning, is to find a collision-free path amidst obstacles for a robot from its starting position to its destination.

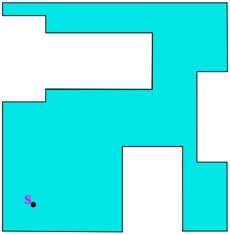


1. J-C Latombe, *Robot Motion Planning*, Kluwer Academic Publishers, Boston, 1991.

Computing configuration space



Work Space



Configuration Space

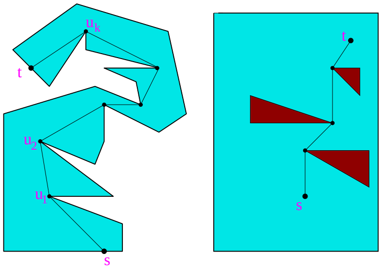
The configuration space can be computed using Minkowski sum.

The problem of computing collision-free path of a rectangle in the actual space is now reduced to that of a point in the free configuration space.

1. T. Lozano-Perez and M. A. Wesley, *An algorithm for planning collision-free paths among polyhedral obstacles*, Communication of ACM, 22 (1979), 560-570.

Computing Euclidean shortest paths

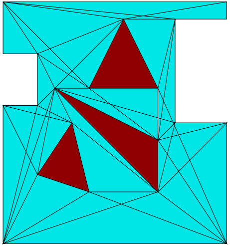
The *Euclidean shortest path* (denoted as $SP(s,t)$) between two points s and t in a polygon P is the path of smallest length between s and t lying totally inside P.

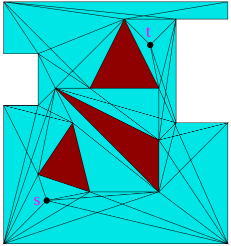


Let $SP(s,t) = (s,\ u_1,\ u_2, \ldots,\ u_k,\ t)$. Then, (i) $SP(s,t)$ is a simple path, (ii) u_1, $u_2, \ldots$, u_k are vertices of P and (iii) for all i, u_i and u_{i+1} are mutually visible in P. $SP(s,t)$ is *outward convex* at every vertex on the path.

Computing $SP(s, t)$ using visibility graph

The visibility graph of a polygon P with non-intersecting polygonal holes or obstacles is a graph whose vertex set consists of the vertices of P and whose edges are visible pairs of vertices.

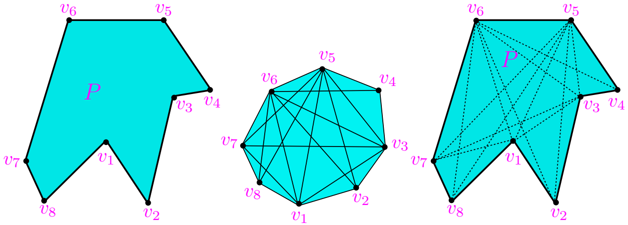




Assign the length of each visible pair as an weight on the corresponding edge in the visibility graph and use the following algorithm to compute $SP(s, t)$.

1. M. L. Fredman and R. E. Tarjan, *Fibonacci heaps and their uses in improved network optimization algorithms*, Journal of ACM, 34 (1987), 596-615. Running time: $O(n \log n + E)$, where E is the number of edges in the visibility graph.
2. S. K. Ghosh and D. M. Mount, *An output sensitive algorithm for computing visibility graphs*, SIAM Journal on Computing, 20 (1991), 888-910. Running time: $O(n \log n + E)$. Space: $O(E)$.
3. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, United Kingdom, 2007.

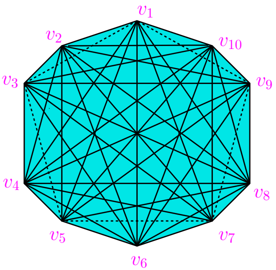
Visibility graphs recognition problem

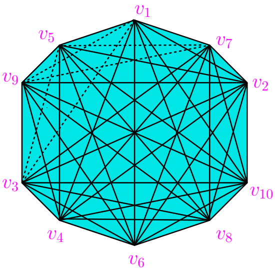


Let G be a graph. The problem of determining, if there is some polygon P that has G as its visibility graph, is called the *visibility graph recognition problem*.

Observe that the boundary of P corresponds to a Hamiltonian cycle of G.

The given graph G can have several Hamiltonian cycles.





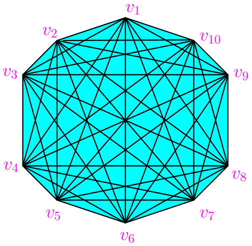
The given graph G is not the visibility graph of any polygon P if the boundary of P corresponds to the Hamiltonian cycle $(v_1, v_5, v_9, v_3, v_4, v_6, v_8, v_{10}, v_2, v_7)$.

Problem: Given an undirected graph G with a Hamiltonian cycle, the problem of recognizing visibility graphs is to test whether there exists a simple polygon such that

1. the Hamiltonian cycle of the graph forms the boundary of a simple polygon,
2. two vertices of the simple polygon are visible if and only if they correspond to adjacent vertices in the graph.

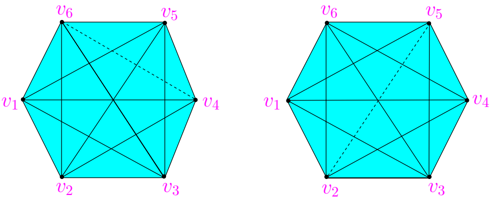
Ghosh's necessary conditions

A cycle $u_1, u_2, \ldots, u_k$ in G is said to be *ordered* if $u_1, u_2, \ldots, u_k$ preserve their order in the Hamiltonian cycle. The Hamiltonian cycle is the longest ordered cycle in G.



Necessary Condition 1. *In a visibility graph, every ordered cycle of $k \geq 4$ vertices has at least $k - 3$ chords.*

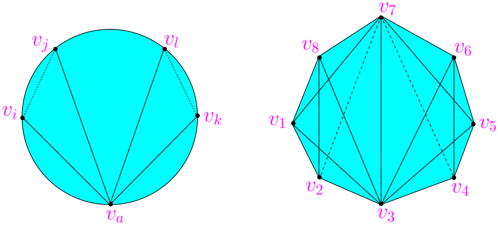
This condition is easily visualized: an ordered cycle corresponds to a sub-polygon and k-vertex sub-polygon must have a triangulation, which has $k - 3$ diagonals.



A vertex v_a is a *blocking vertex* for an invisible pair (v_i, v_j) if no two vertices $k \in chain(v_i, v_{a-1})$ and $m \in chain(v_{a+1}, v_j)$ are adjacent in G.

The vertex v_a is called a blocking vertex because v_a can be used to block the line of sight between v_i and v_j in the polygon.

Necessary Condition 2. *In a visibility graph, every invisible pair has at least one blocking vertex.*



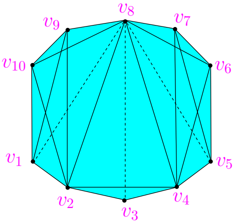
Two invisible pairs (v_i, v_j) and (v_k, v_l) are called *separable* with respect to a vertex v_a if v_k and v_l are encountered before v_i and v_j when the Hamiltonian cycle is traversed from v_a.

Necessary Condition 3. *In a visibility graph, two separable invisible pairs must have distinct blocking vertices.*

Ghosh conjectured in 1986 that the three necessary conditions are sufficient.

1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Report JHU/E86/14, The Johns Hopkins University, 1986. Also in Lecture Notes in Computer Science, Springer-Verlag, pp. 96-104, 1988.

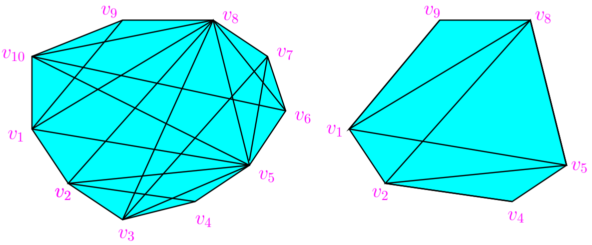
Counter-examples



The vertex v_2 can block either (v_1, v_8) or (v_3, v_8). The vertex v_4 can block either (v_3, v_8) or (v_5, v_8).

Necessary Condition 3′. *In a visibility graph, there is an assignment such that no blocking vertex v_a is assigned to two or more minimal invisible pairs that are separable with respect to v_a.*

1. H. Everett, *Visibility graph recognition*, Ph. D. Thesis, University of Toronto, 1990.
2. G. Srinivasaraghavan and A. Mukhopadhyay, *A new necessary condition for the vertex visibility graphs of simple polygons*, Discrete and Computational Geometry, 12 (1994), 65–82.



Four vertices v_1, v_2, v_5, v_8 are reflex vertices in six vertices subpolygon $(v_1, v_2, v_4, v_5, v_8, v_9, v_1)$ since v_1, v_2, v_5, v_8 are assigned to invisible pairs (v_2, v_9), (v_1, v_4), (v_4, v_8) and (v_5, v_9) respectively.

1. J. Abello, K. Kumar and S. Pisupati, *On visibility graphs of simple polygons*, Congressus Numeratium, 90 (1992), 119-128.

Modified conjecture

Necessary Condition 4. Let D be any ordered cycle of a visibility graph. For any assignment of blocking vertices to all minimal invisible pairs in the visibility graph, the total number of vertices of D assigned to the minimal invisible pairs between the vertices of D is at most $|D| - 3$.

Ghosh's Conjecture: These four conditions $(1, 2, 3'$, and $4)$ are not only necessary, but also sufficient.

1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.

Testing necessary conditions

Everett presented an $O(n^3)$ time algorithm for testing Necessary Condition 1 which was later improved by Ghosh to $O(n^2)$ time.

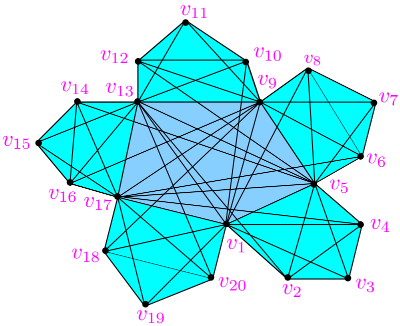
Ghosh also gave an $O(n^2)$ time algorithm for testing Necessary Condition 2.

Das et al. showed that Necessary Condition 3 can be tested in $O(n^4)$ time.

It is open whether Necessary Condition 4 can be tested in polynomial time.

1. H. Everett, *Visibility graph recognition*, Ph. D. Thesis, University of Toronto, 1990.
2. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.
3. S. Das, P. Goswami, S. Nandy, *Testing necessary conditions for recognizing visibility graphs of simple polygons*, Manuscript, Indian Statistical Institute, 2002.

Another counter-example



This graph satisfies all four necessary conditions but it is not the visibility graph of any simple polygon.

The graph is given by Streinu as a counter-example to Ghosh's new conjecture of sufficiency.

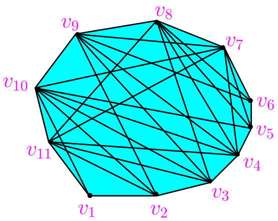
1. I. Streinu, *Non-stretchable pseudo-visibility graphs*, Computational Geometry: Theory and Applications, 31 (2005), 195-206.

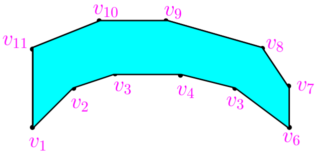
It is not clear whether either another necessary condition is required to circumvent this counter-example or there is a need to strengthen the existing necessary conditions.

We hope that the visibility graph recognition problem will be settled in the near future.

1. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, United Kingdom, 2007.

Recognizing spiral polygons

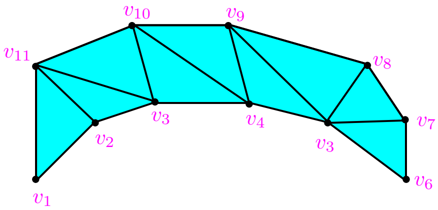




A simple polygon is said to be *spiral* if its boundary consists of one chain of reflex vertices and one chain of convex vertices.

A vertex is *simplicial* if its neighborhood is a clique. There always exists two *simplicial* vertices (here, vertices v_1 and v_6) in the visibility graph of a spiral polygon.

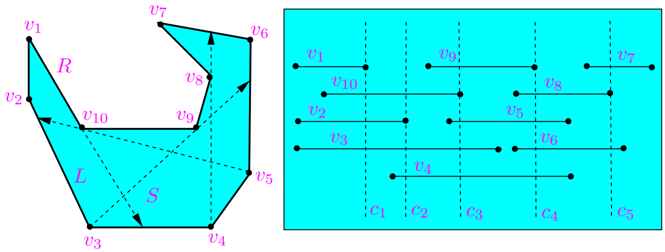
1. H. Everett and D. Corneil, *Recognizing visibility graphs of spiral polygons*, Journal of Algorithms, 11 (1990), 1-26.



Once a simplicial vertex is eliminated, another vertex in the remaining graph becomes simplicial. Here, the sequence of simplicial vertices is $(v_1, v_2, v_{11}, v_3, v_{10}, v_4, v_9, v_8, v_3, v_7, v_6)$.

Sequence of simplicial vertices forms a perfect vertex elimination scheme.

A graph is a *chordal* if and only if it has a perfect vertex elimination scheme. So, visibility graphs of spiral polygons are chordal.



Three non-adjacent vertices are called an *asteroidal triple* if they they cannot be ordered is such a way that every path from first vertex to the third vertex passes through a neighbor of the second vertex.

A graph is an *interval graph* if and only if the graph is a chordal graph containing no asteroidal triples.

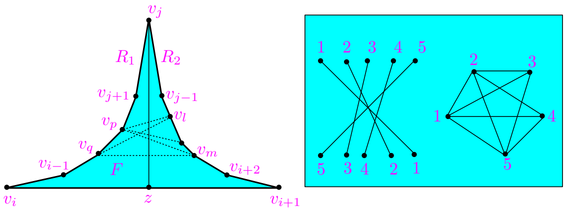
Visibility graphs of spiral polygons do not contain any asteroidal triples and therefore, they are interval graphs.

A graph G is the visibility graph of a spiral polygon if and only if
(i) G is an interval graph, and
(ii) the paths corresponding to L and R formed by conductors in the ordering of maximal cliques in G satisfy some geometric properties.

Visibility graphs of spiral polygons can be recognized in $O(n)$ time.

1. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, United Kingdom, 2007.

Recognizing tower polygons



A *tower* polygon F is a simple polygon formed by two reflex chains of vertices with only one boundary edge connecting two convex vertices.

Removing apex vertex v_j and boundary edges of F from the visibility graph G of F to form the cross-visible sub-graph G' of G.

G' is a bipartite permutation graph as it satisfies strong ordering.

A given graph G is the visibility graph of a tower polygon if and only if its cross-visible sub-graph G' is a bipartite permutation graph.

It takes $O(n)$ time to test whether G' is a bipartite permutation graph.

A Hamiltonian cycle in G which corresponds to the boundary of a tower polygon can be constructed in $O(n)$ time.

Visibility graphs of tower polygons can be recognized in $O(n)$ time.

1. P. Colley, A. Lubiw and J. Spinrad, *Visibility graphs of towers*, Computational Geometry: Theory and Applications, 7 (1997), 161-172.
2. S.-H. Choi, S. Y. Shin and K.-Y. Chwa, *Characterizing and recognizing the visibility graph of a funnel-shaped polygon*, Journal of Algorithms, 14 (1995), 27-51.

Characterizing visibility graphs

Characterizing visibility graphs of simple polygons is an open problem.

Ghosh tried to characterize visibility graphs in term of perfect graphs, circle graphs or chordal graphs.

Coullard and Lubiw used clique ordering properties in their attempt to characterize visibility graphs.

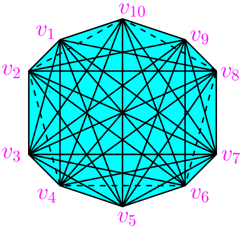
1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Report JHU/E86/14, The Johns Hopkins University, 1986. Also in Lecture Notes in Computer Science, Springer-Verlag, pp. 96-104, 1988.
2. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.
3. C. Coullard and A. Lubiw, *Distance Visibility graphs*, International Journal of Computational Geometry and Applications, 2 (1992), 349–362.

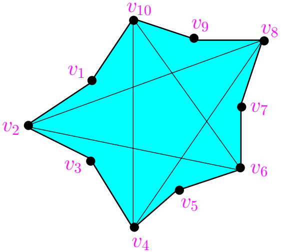
Everett and Corneil attempted to characterize visibility graphs using forbidden induced sub-graphs.

Abello and Kumar attempted to characterize visibility graphs using Euclidean shortest paths.

1. H. Everett, *Visibility graph recognition*, Ph. D. Thesis, University of Toronto, 1990.
2. H. Everett and D. G. Corneil, *Negative results on characterizing visibility graphs*, Computational Geometry: Theory and Applications, 5 (1995), 51-63.
3. J. Abello and K. Kumar, *Visibility Graphs and Oriented Matroids*, Discrete and Computational Geometry, 28 (2002), 449-465.

Is visibility graph a perfect graph?



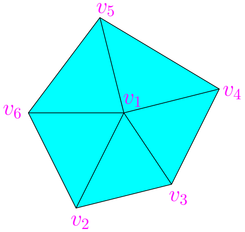


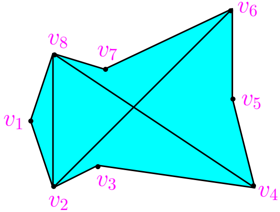
Ghosh's Necessary Condition 1 suggests that every ordered cycle in a visibility graph has a chord. Is it true that every unordered cycle in a visibility graph has also a chord?

In the above figure, vertices v_2, v_8, v_4, v_{10} and v_6 form an odd cycle without a diagonal.

Therefore, the chromatic number is not equal to the maximum cardinality clique in the visibility graph. Hence the visibility graph of a simple polygon is not a perfect graph (as well as chordal graph).

Is visibility graph a circle graph?





An undirected graph G is called *circle graph* if there exists a set of chords C on a circle and one-to-one correspondence between vertices of G and chords of C such that two distinct vertices are adjacent in G if and only if their corresponding chords intersect.

A graph G is not a circle graph if G contains a wheel. Vertex v_1 and the cycle v_2, v_6, v_5, v_4 and v_8 have formed a wheel.

Visibility graphs of simple polygons do not belong to the union of perfect graphs and circle graphs.

We feel that visibility graphs of simple polygons in general form a new class of graphs. However, the possibility of characterizing visibility graphs in terms of any known class of graphs cannot be totally ruled out.

1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.
2. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, United Kingdom, 2007.

Concluding remarks

In-spite of several attempts by many researchers in the last three decades, the problems of recognizing, characterizing and reconstructing visibility graphs of simple polygons are still open.