Matching Theory

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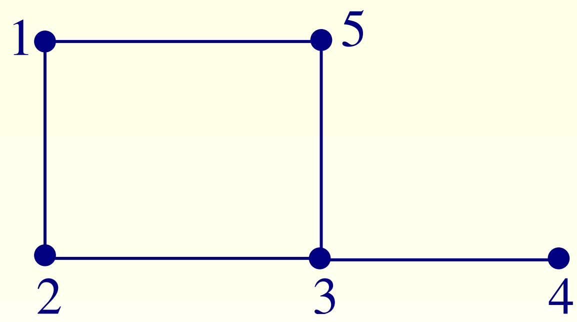
Definition: A simple graph is a triple $G = (V, E, \phi)$, where V is a finite set of vertices, E is a finite set of edges, and ϕ is a function that assigns to each edge $e \in E$ a 2-element set of vertices. Thus $\phi : E \to \binom{V}{2}$ and ϕ is injective (one-to-one).

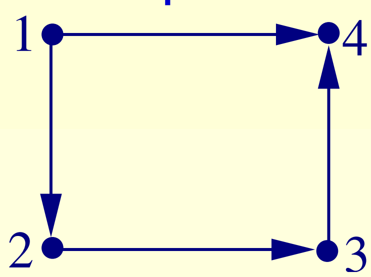
Example:

$V = \{1, 2, 3, 4, 5\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $\phi(e_1) = \{1, 2\}$, $\phi(e_2) = \{2, 3\}$, $\phi(e_3) = \{3, 4\}$, $\phi(e_4) = \{5, 1\}$, $\phi(e_5) = \{3, 5\}$.



In a directed graph $\phi : E \to V \times V - \Delta$ and injective.

Example:



Definition: A set of edges in a simple graph $G = (V, E)$ is called a matching if no two edges have a vertex in common. ($M \subset E(G)$ and $e_1, e_2 \in M$ implies $e_1 \cap e_2 = \emptyset$.)

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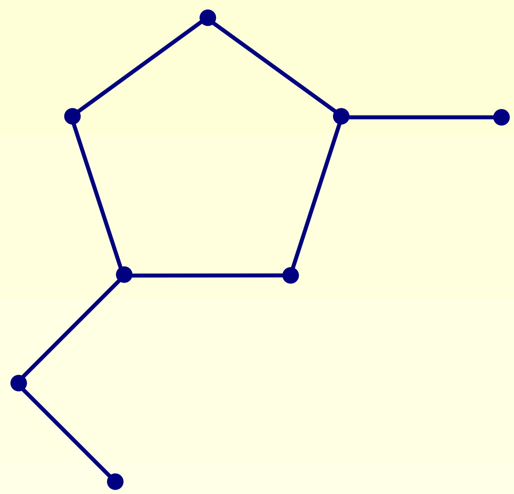

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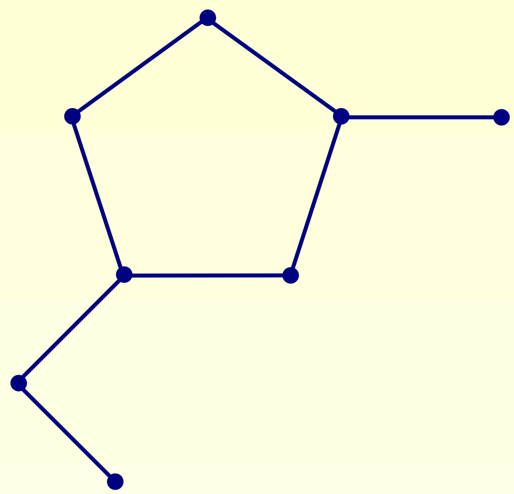

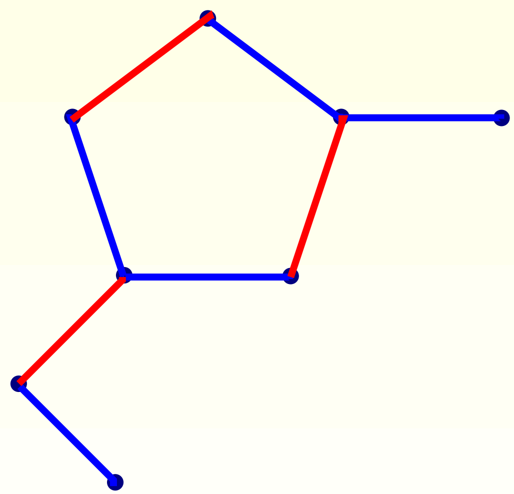

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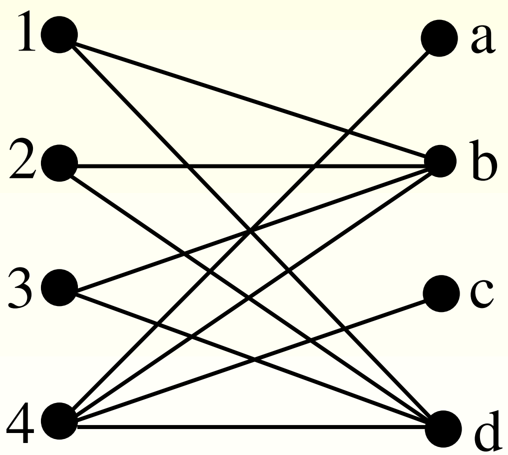

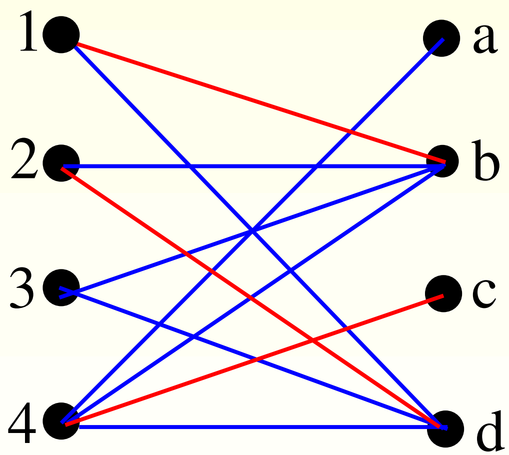

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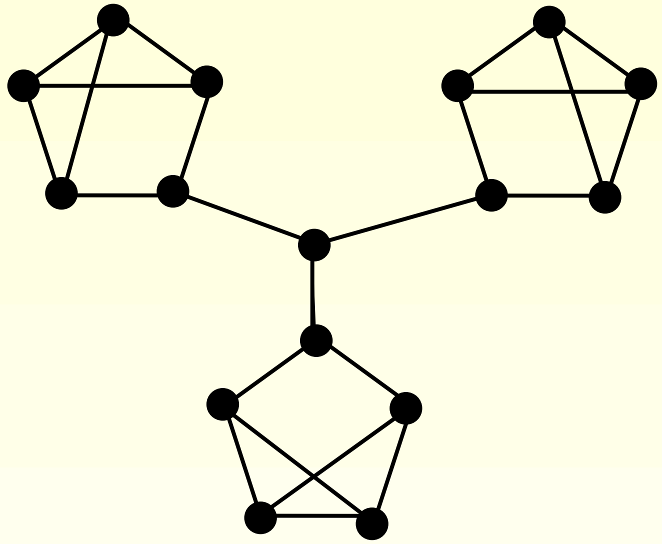
Matching in nonbipartite graphs.



Theorem (Tutte) A graph $G(V, E)$ has a perfect matching iff $\forall X \subset V$, $|X| \geq q_G(X)$.

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Tutte Matrix:

Orient the edges in $G(V, E)$ to obtain $D(V, E)$.

$$(T_G(\mathbf{x}))_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in E(D), \\ -x_{i,j} & \text{if } (j,i) \in E(D) \\ 0 & \text{otherwise.} \end{cases}$$

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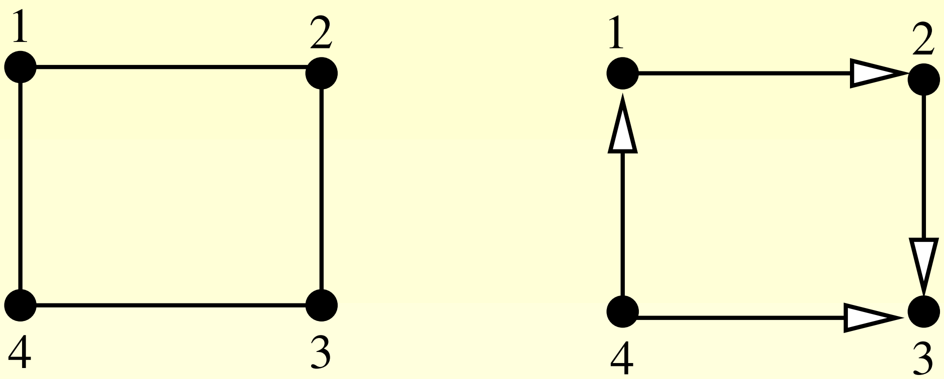
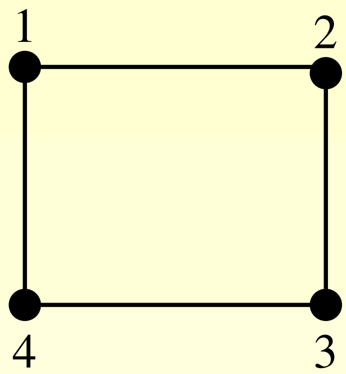
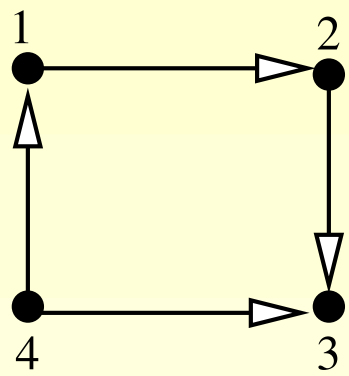


$$T_G(\mathbf{x}) = \begin{bmatrix} 0 & x_{12} & 0 & -x_{14} \\ -x_{12} & 0 & x_{23} & 0 \\ 0 & -x_{23} & 0 & -x_{34} \\ x_{14} & 0 & x_{34} & 0 \end{bmatrix}$$





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$$Det(T_G(\mathbf{x})) = x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2 + 2x_{12}x_{14}x_{23}x_{34}.$$

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Corollary: (Lovasz)

Let $\mathbf{x}$ be a random vector where each coordinate is uniformly distributed in $[0, 1]$. Then with probability 1 the rank of $T_G(\mathbf{x})$ is exactly twice the size of a maximum matching.

Algorithms for matching.

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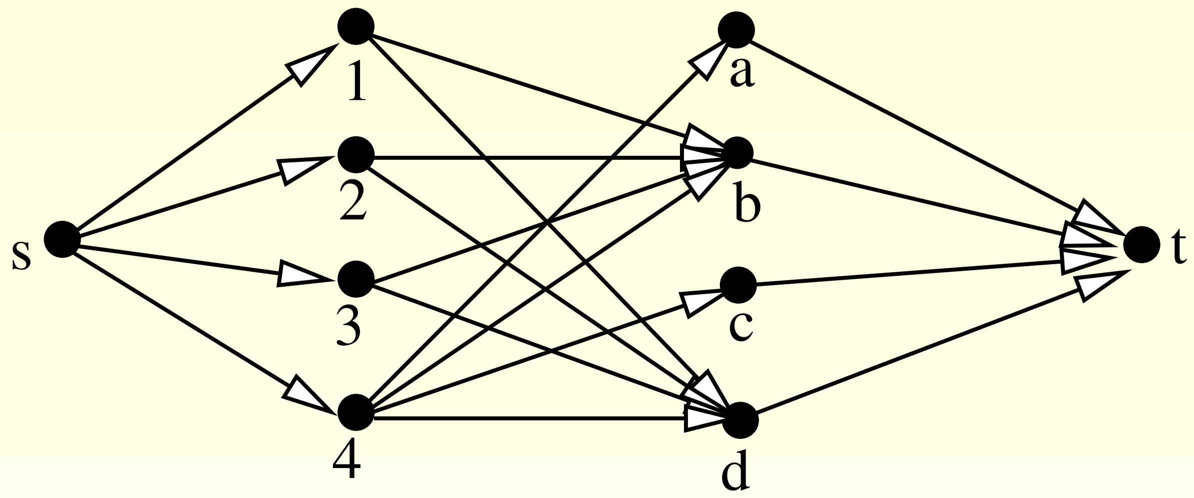

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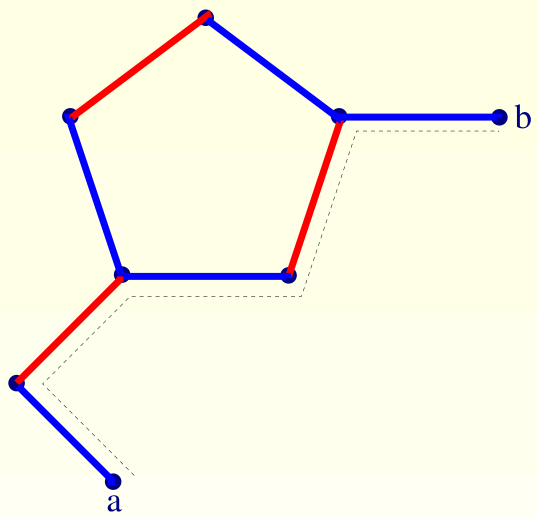

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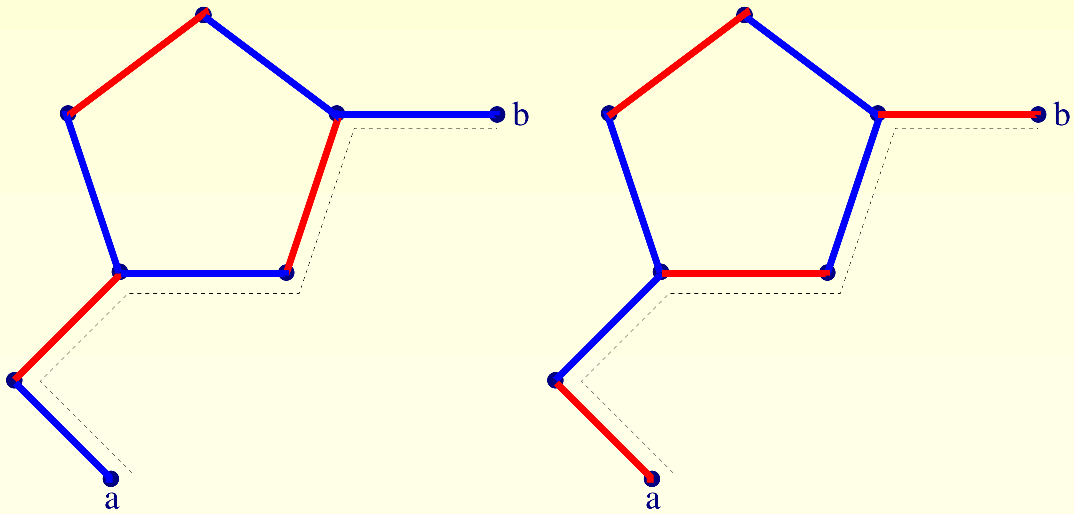

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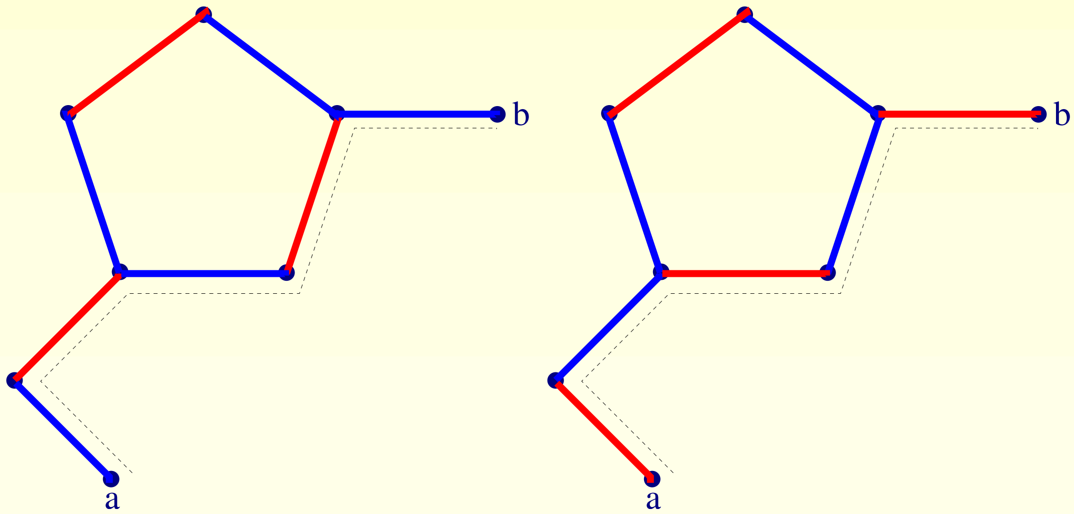

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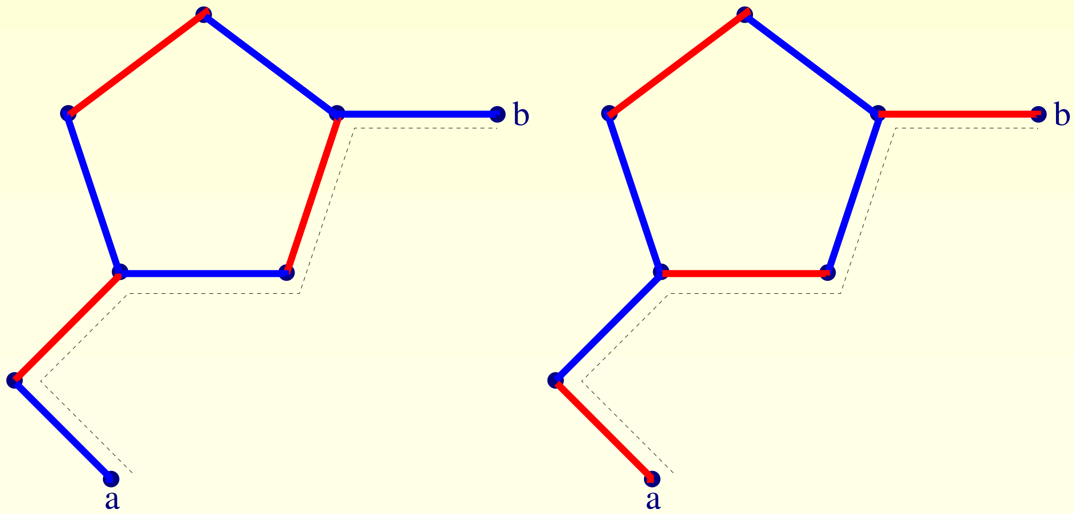
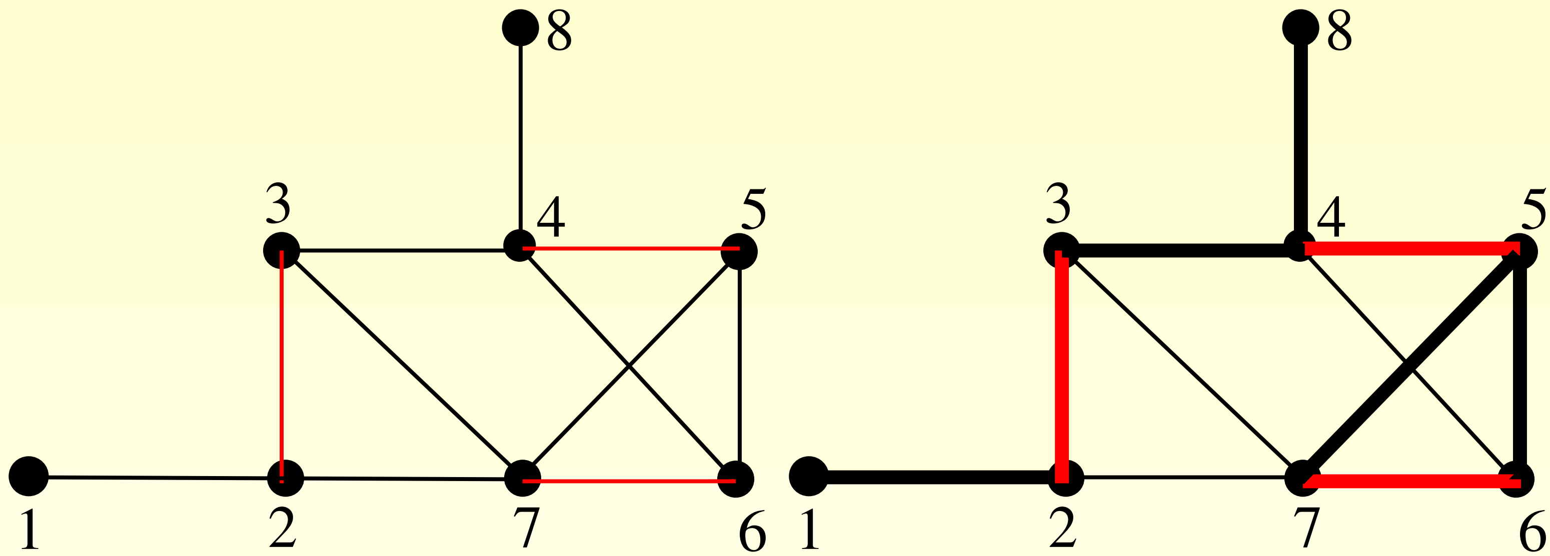


In bipartite case finding augmenting paths is “easy”.

In nonbipartite case finding augmenting paths is tricky.



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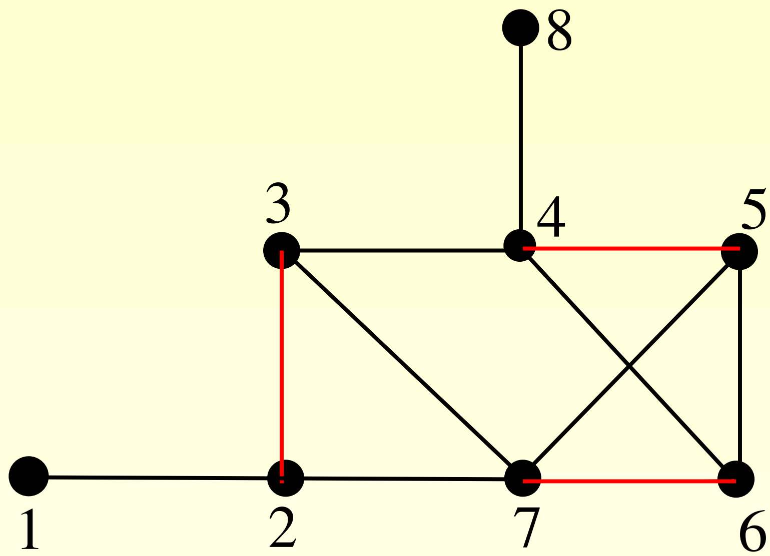
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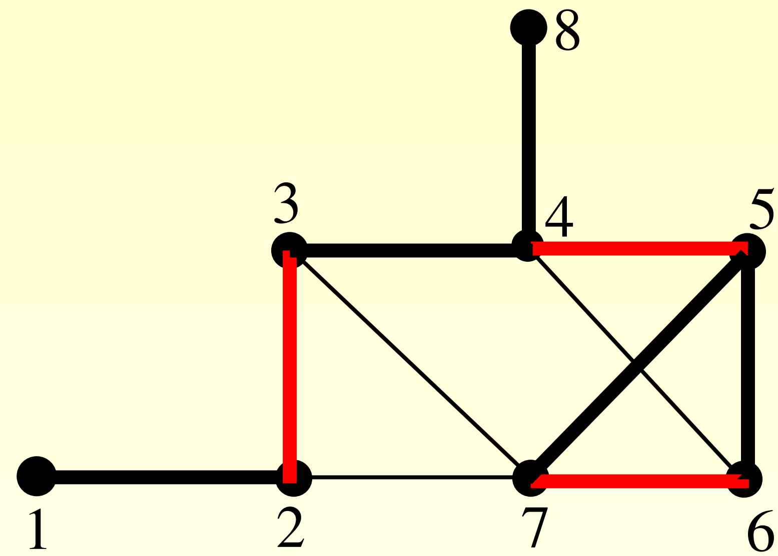
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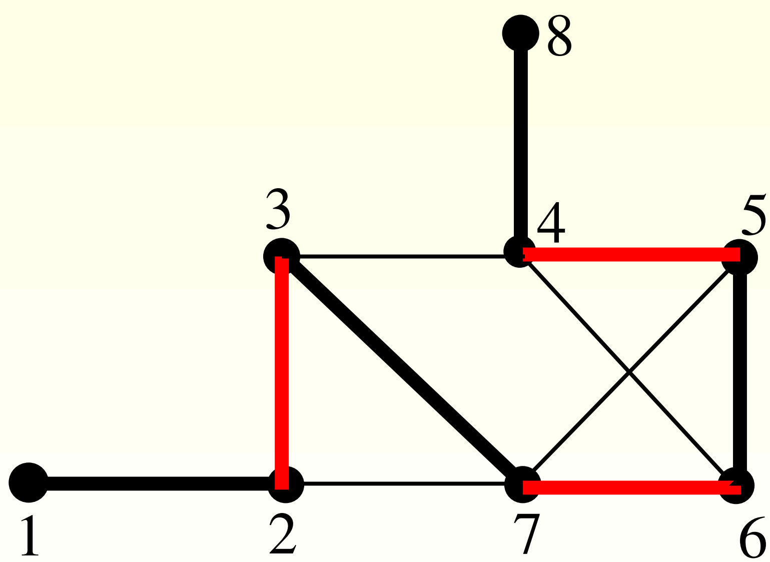
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Efficient implementation requires good data structures.

Running time $O(|E|\sqrt{|V|}\frac{\log\frac{|V|^2}{|E|}}{\log|V|})$.

Application:

Problem (Chinese postman Problem)

Let $G(V, E)$ be a graph with $w : E(G) \longrightarrow \mathbb{R}_{\geq 0}$. Find $f : E(G) \longrightarrow \mathbb{N}$ such that the (need not be simple) graph which arises from G by repeating each edge e, $f(e)$ times is Eulerian and $\sum_{e \in E(G)} f(e)w(e)$ is minimum.

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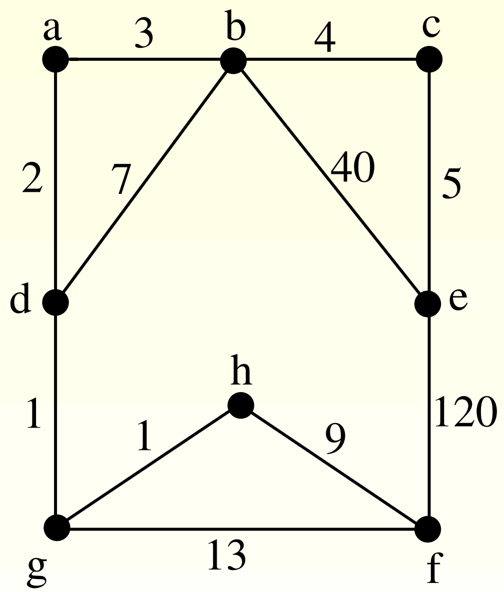

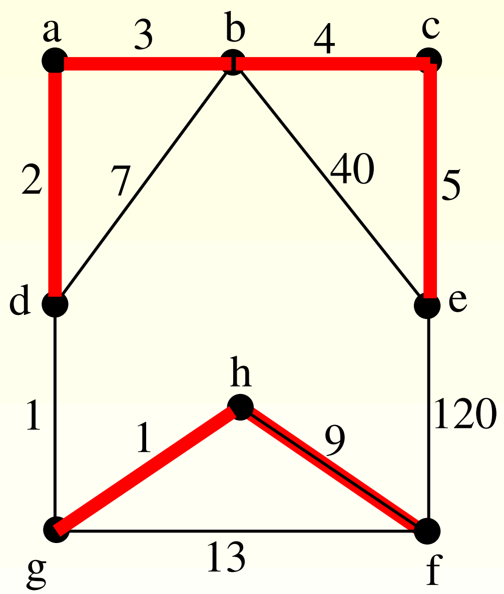

Open problems:

Conjecture (Berge-Fulkerson) If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.

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Conjecture: (Seymour) Let G be an r-graph. Then there exist $2r$ perfect matchings $M_1, \ldots, M_{2r}$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_{2r}$.

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Conjecture: There exists a fixed integer k such that every 3-graph has a list of k perfect matchings with empty intersection.

Thank You