Introduction to Randomized Algorithms

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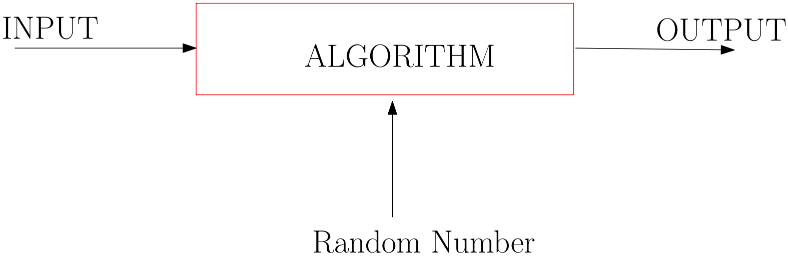
Organization

Introduction

Goal of a Deterministic Algorithm

- The solution produced by the algorithm is correct, and
- the number of computational steps is same for different runs of the algorithm with the same input.

Randomized Algorithm



Randomized Algorithm

- In addition to the input, the algorithm uses a source of pseudo random numbers. During execution, it takes random choices depending on those random numbers.
- The behavior (output) can vary if the algorithm is run multiple times on the same input.

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The Paradigm

Instead of making a guaranteed good choice, make a random choice and hope that it is good. This helps because guaranteeing a good choice becomes difficult sometimes.

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Average Case Analysis

analyzes the expected running time of deterministic algorithms assuming a suitable random distribution on the input.

Pros and Cons of Randomized Algorithms

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- There is a finite probability of getting incorrect answer. However, the probability of getting a wrong answer can be made arbitrarily small by the repeated employment of randomness.
- Getting true random numbers is almost impossible.

Types of Randomized Algorithms

Definition

Las Vegas: a randomized algorithm that always returns a correct result. But the running time may vary between executions.

Example: Randomized QUICKSORT Algorithm

Definition

Monte Carlo: a randomized algorithm that terminates in polynomial time, but might produce erroneous result.

Example: Randomized MINCUT Algorithm

Some basic ideas from Probability

Expectation

Random variable

A function defined on a sample space is called a random variable. Given a random variable X, $Pr[X = j]$ means X's probability of taking the value j.

Expectation – "the average value"

The expectation of a random variable X is defined as:
$E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j]$

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- So, the expectation of X, $E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j] = \frac{1}{p}$.

Conditional Probability and Independent Event

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The conditional probability of X given Y is

$$Pr[X = x \mid Y = y] = \frac{Pr[(X = x) \cap (Y = y)]}{Pr[Y = y]}$$

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Independent Events

Two events X and Y are independent, if $Pr[(X = x) \cap (Y = y)] = Pr[X = x] \cdot Pr[Y = y]$. In particular, if X and Y are independent, then

$$Pr[X = x \mid Y = y] = Pr[X = x]$$

A Result on Intersection of events

Let $\eta_1, \eta_2, \ldots, \eta_n$ be n events not necessarily independent. Then,

$$Pr[\cap_{i=1}^{n}\eta_i] = Pr[\eta_1] \cdot Pr[\eta_2 \mid \eta_1] \cdot Pr[\eta_3 \mid \eta_1 \cap \eta_2] \cdots Pr[\eta_n \mid \eta_1 \cap \ldots \cap \eta_{n-1}].$$

The proof is by induction on n.

Coupon Collection

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A company selling jeans gives a coupon with each jeans. There are n different coupons. Collecting n different coupons would give you a free jeans. How many jeans do you expect to buy before you get a free jeans?

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- The coupon collection process is in phase j when you have already collected j different coupons and are buying to get a new type.
- A new type of coupon ends phase j and you enter phase $j+1$.

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Lemma

The expected number of jeans bought in phase j, $E[X_j] = \frac{n}{n-j}$.

- The success probability, p in the j-th phase is $\frac{n-j}{n}$.
- By the bound on waiting for success, the expected number of jeans bought $E[X_j]$ is $\frac{1}{p} = \frac{n}{n-j}$.

The expectation

Theorem

The expected number of jeans bought before all n types of coupons are collected is $E[X] = nH_n = \Theta(n \log n)$.

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Proof

$X = \sum_{j=0}^{n-1} X_j$. So, we have $E[X] = E\left[\sum_{j=0}^{n-1} X_j\right]$. Use linearity of expectations,

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = n \sum_{j=0}^{n-1} \frac{1}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = nH_n = \Theta(n \log n)$$

Randomized Quick Sort

Deterministic Quick Sort

The Problem:

Given an array $A[1 \ldots n]$ containing n (comparable) elements, sort them in increasing/decreasing order.

QSORT(A, p, q)

- If $p \geq q$, EXIT.
- Compute $s \leftarrow$ correct position of $A[p]$ in the sorted order of the elements of A from p-th location to q-th location.
- Move the pivot $A[p]$ into position $A[s]$.
- Move the remaining elements of $A[p - q]$ into appropriate sides.
- QSORT(A, p, $s - 1$);
- QSORT(A, $s + 1$, q).

Complexity Results of QSORT

- An INPLACE algorithm
- The worst case time complexity is $O(n^2)$.
- The average case time complexity is $O(n \log n)$.

Randomized Quick Sort

An Useful Concept - The Central Splitter

It is an index s such that the number of elements less (resp. greater) than $A[s]$ is at least $\frac{n}{4}$.

- The algorithm randomly chooses a key, and checks whether it is a central splitter or not.
- If it is a central splitter, then the array is split with that key as was done in the QSORT algorithm.
- It can be shown that the expected number of trials needed to get a central splitter is constant.

Randomized Quick Sort

RandQSORT(A, p, q)

1: If $p \geq q$, then EXIT.

2: While no central splitter has been found, execute the following steps:

  2.1: Choose uniformly at random a number $r \in \{p, p+1, \ldots, q\}$.

  2.2: Compute s = number of elements in A that are less than $A[r]$, and
  t = number of elements in A that are greater than $A[r]$.

  2.3: If $s \geq \frac{q-p}{4}$ and $t \geq \frac{q-p}{4}$, then $A[r]$ is a central splitter.

3: Position $A[r]$ in $A[s+1]$, put the members in A that are smaller than the central splitter in $A[p \ldots s]$ and the members in A that are larger than the central splitter in $A[s+2 \ldots q]$.

4: RandQSORT(A, p, s);

5: RandQSORT(A, $s+2$, q).

Analysis of RandQSORT

Fact: One execution of Step 2 needs $O(q - p)$ time.

Question: How many times Step 2 is executed for finding a central splitter ?

Result:

The probability that the randomly chosen element is a central splitter is $\frac{1}{2}$.

Recall "Waiting for success"

If p be the probability of success of a random experiment, and we continue the random experiment till we get success, the expected number of experiments we need to perform is $\frac{1}{p}$.

Implication in Our Case

- The expected number of times Step 2 needs to be repeated to get a central splitter (success) is 2 as the corresponding success probability is $\frac{1}{2}$.
- Thus, the expected time complexity of Step 2 is $O(n)$

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- By linearity of expectations, the expected time for all partitions of size $n \cdot (\frac{3}{4})^j$ is $O(n)$.

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- Number of levels of recursion $= \log_{\frac{4}{3}} n = O(\log n)$.
- Thus, the expected running time is $O(n \log n)$.

Finding the k-th largest

Median Finding

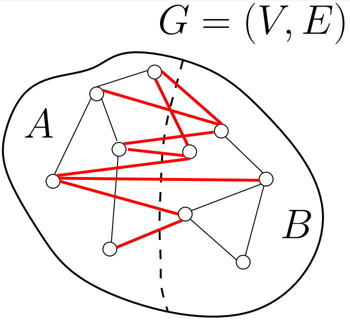
Similar ideas of getting a central splitter and waiting for success bound applies for finding the median in $O(n)$ time.

Global Mincut Problem for an Undirected Graph

Global Mincut Problem

Problem Statement

Given a connected undirected graph $G = (V, E)$, find a cut (A, B) of minimum cardinality.



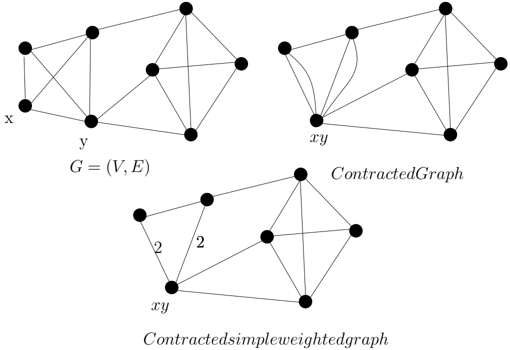
Applications:

- Clustering and partitioning items,
- Network reliability, network design, circuit design, etc.

A Simple Randomized Algorithm

Contraction of an Edge

Contraction of an edge $e = (x, y)$ implies merging the two vertices $x, y \in V$ into a single vertex, and remove the self loop. The contracted graph is denoted by G/xy.



Results on Contraction of Edges

Result - 1

As long as G/xy has at least one edge,

- The size of the minimum cut in the (weighted) graph G/xy is at least as large as the size of the minimum cut in G .

Result - 2

Let $e_1, e_2, \ldots, e_{n-2}$ be a sequence of edges in G, such that

- none of them is in the minimum cut of G, and
- $G' = G/\{e_1, e_2, \ldots, e_{n-2}\}$ is a single multiedge.

Then this multiedge corresponds to the minimum cut in G.

Problem: Which edge sequence is to be chosen for contraction?

Analysis

Algorithm MINCUT(G)

$G_0 \leftarrow G$; $i = 0$
while G_i has more than two vertices **do**
  Pick randomly an edge e_i from the edges in G_i
  $G_{i+1} \leftarrow G_i / e_i$
  $i \leftarrow i + 1$
$(S, V - S)$ is the cut in the original graph
  corresponding to the single edge in G_i.

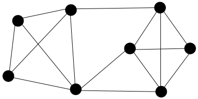
Theorem

Time Complexity: $O(n^2)$

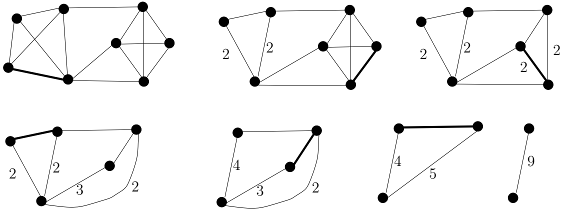
A Trivial Observation: The algorithm outputs a cut whose size is *no smaller than the mincut*.

Demonstration of the Algorithm

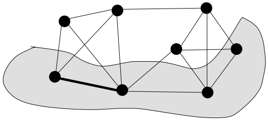
The given graph:



Stages of Contraction:



The corresponding output:



Quality Analysis: How good is the solution?

Result 3: Lower bounding $|E|$

If a graph $G = (V, E)$ has a minimum cut F of size k, and it has n vertices, then $|E| \geq \frac{kn}{2}$.

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- Thus, every node in G has degree at least k. So, $|E| \geq \frac{1}{2}kn$.

So, the probability that an edge in F is contracted is at most
$\frac{k}{(kn)/2} = \frac{2}{n}$
But, we don't know the min cut.

Summing up: Result 4

If we pick a random edge e from the graph G, then the probability of e belonging in the mincut is at most $\frac{2}{n}$.

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- Thus, G' has at least $\frac{1}{2}k(n-i)$ edges.
- So, the probability that an edge in F is contracted in iteration $i+1$ is at most $\frac{k}{\frac{1}{2}k(n-i)} = \frac{2}{n-i}$.

Correctness

Theorem

The procedure MINCUT outputs the mincut with probability $\geq \frac{2}{n(n-1)}$.

Proof:

The correct cut(A, B) will be returned by MINCUT if no edge of F is contracted in any of the iterations $1, 2, \ldots, n-2$.

Let $\eta_i \Rightarrow$ the event that an edge of F is not contracted in the ith iteration.

We have already shown that

- $Pr[\eta_1] \geq 1 - \frac{2}{n}$.
- $Pr[\eta_{i+1} \mid \eta_1 \cap \eta_2 \cap \cdots \cap \eta_i] \geq 1 - \frac{2}{n-i}$

Lower Bounding the Intersection of Events

We want to lower bound $Pr[\eta_1 \cap \cdots \cap \eta_{n-2}]$.
We use the earlier result

$$Pr[\cap_{i=1}^{n} \eta_i] = Pr[\eta_1] \cdot Pr[\eta_2 \mid \eta_1] \cdot Pr[\eta_3 \mid \eta_1 \cap \eta_2] \cdots Pr[\eta_n \mid \eta_1 \cap \ldots \cap \eta_{n-1}].$$

So, we have $Pr[\eta_1] \cdot Pr[\eta_1 \mid \eta_2] \cdots Pr[\eta_{n-2} \mid \eta_1 \cap \eta_2 \cdots \cap \eta_{n-3}]$
$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{n-i}\right) \cdots \left(1 - \frac{2}{3}\right)$
$= \binom{n}{2}^{-1}$

Bounding the Error Probability

- We know that a single run of the contraction algorithm fails to find a global min-cut with probability at most $1 - \frac{1}{\binom{n}{2}} \approx 1$.

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- If we run the algorithm $\binom{n}{2}$ times, then the probability that we fail to find a global min-cut in any run is at most

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Result

By spending $O(n^4)$ time, we can reduce the failure probability from $1 - \frac{2}{n^2}$ to a reasonably small constant value $\frac{1}{e}$.

Probability helps in counting

The number of global minimum cuts

Given an undirected graph $G = (V, E)$ with $|V| = n$, what is the maximum number of global minimum cuts?

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Given an undirected graph $G = (V, E)$ with $|V| = n$, what is the maximum number of global minimum cuts?

- What is your hunch? – exponential in n or polynomial in n?

Probability helps in counting

The number of global minimum cuts

Given an undirected graph $G = (V, E)$ with $|V| = n$, what is the maximum number of global minimum cuts?

- What is your hunch? – exponential in n or polynomial in n?
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- Surely, $\Pr[\mathcal{E}] \leq 1$. So, $r \leq \binom{n}{2}$.

Conclusions

- Employing randomness leads to improved simplicity and improved efficiency in solving the problem.
- It assumes the availability of a perfect source of independent and unbiased random bits.
- Access to truly unbiased and independent sequence of random bits is expensive.
  So, it should be considered as an expensive resource like time and space.
- There are ways to reduce the randomness from several algorithms while maintaining the efficiency nearly the same.

Books

- Jon Kleinberg and Éva Tardos, *Algorithm Design*, Pearson Education.
- Rajeev Motwani and Prabhakar Raghavan, *Randomized Algorithms*, Cambridge University Press, Cambridge, UK, 2004.
- Michael Mitzenmacher and Eli Upfal, *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*, Cambridge University Press, New York, USA, 2005..