Introduction to Geometric Data Structures

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Outline

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- Different kinds of data structure are suited to different kinds of application, and some are highly specialized to specific tasks.
- Computational geometry often require preprocessing geometric objects into a simple and space-efficient structure so that the operations on the geometric objects can be performed repeatedly in an efficient manner.

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- In this lecture we introduce a few simple and basic geometric data structures.
- In our discussion, we consider several problems which we want to solve in repetitive query mode. This means, data set is given a priori and we are allowed to preprocess the data-set. Queries come repetitively and we want to answer them efficiently. Various data structures will be introduced whose use lead efficient solution of the problems considered.

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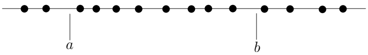
1-DIMENSIONAL RANGE SEARCHING

Problem

Given a set P of n points $p_1, p_2, \ldots, p_n$ *on the real line, report points of P that lie in the range* $[a, b], a \leq b$.



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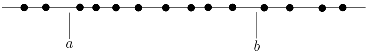

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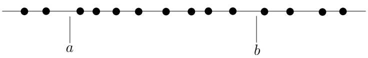

- Using array for storing P, we can use binary search on the array to answer such a query in $O(\log n + k)$ time where k is the number of points of P reported.
- Problems with this solution are that it can't be generalized to higher dimensions and it does not allow for efficient updates.

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- Here is an example.

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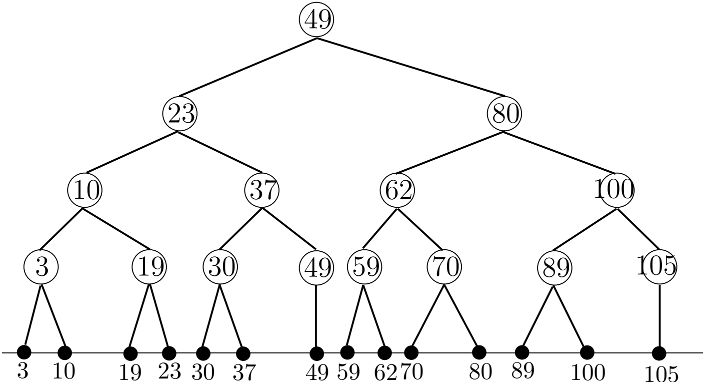

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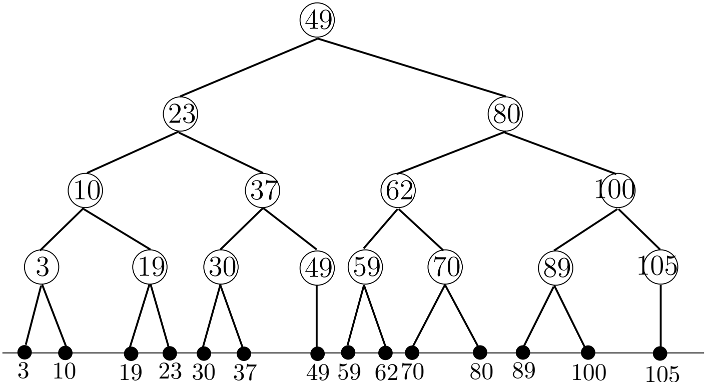

- Such a balanced binary search tree $\mathcal{T}$ on n points can be constructed in $n \log n$ time and it uses n storage.

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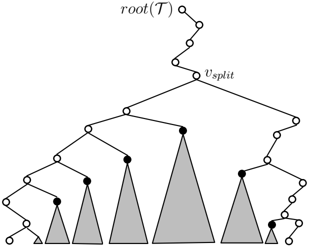

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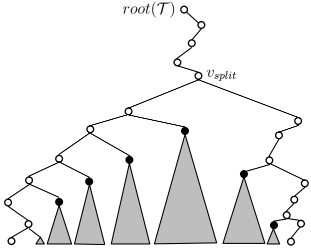

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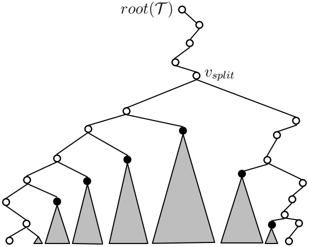

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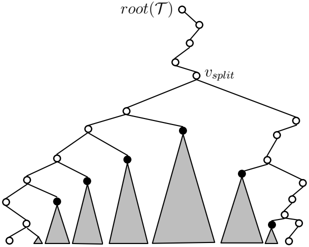
- Finally we check the points stored at the leaves where paths end; we may or may not need to report them.



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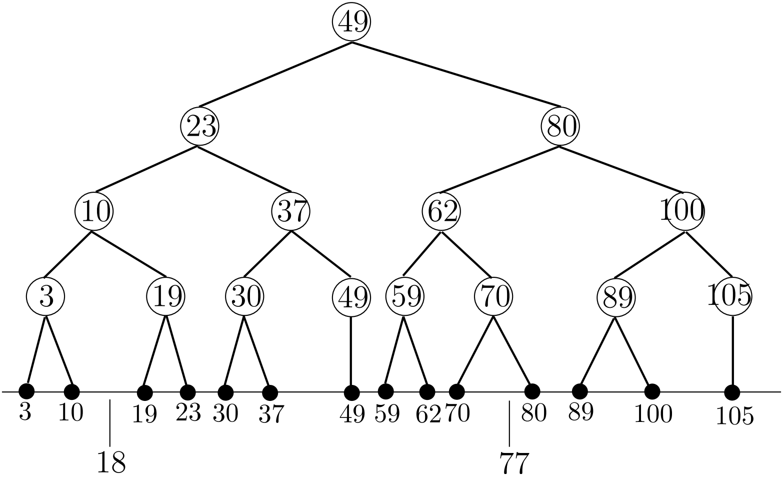

Searching with the query range $[18, 77]$.

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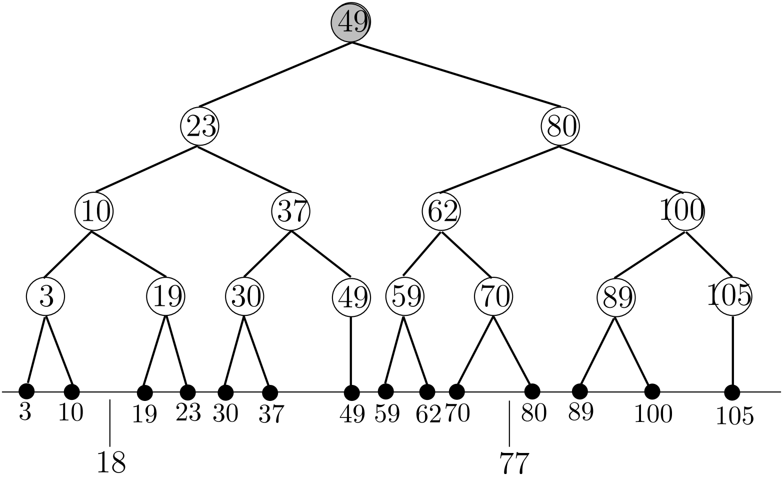

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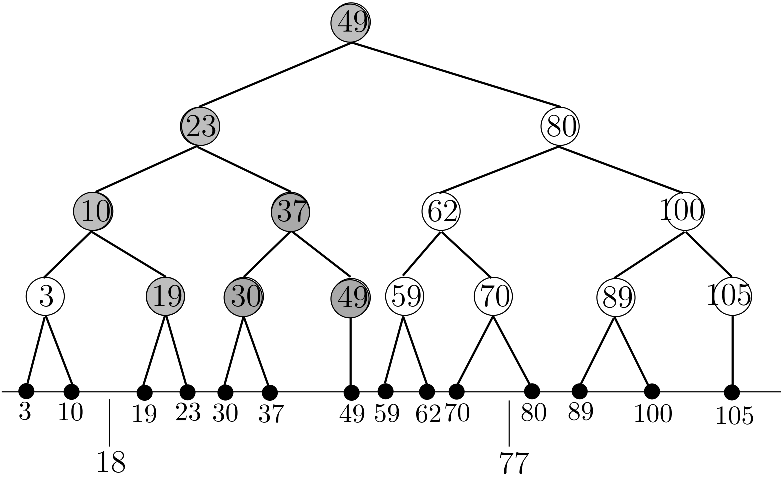

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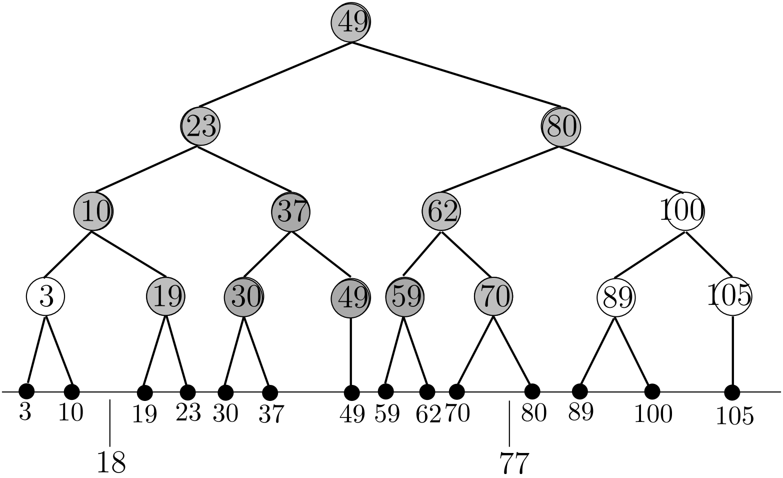

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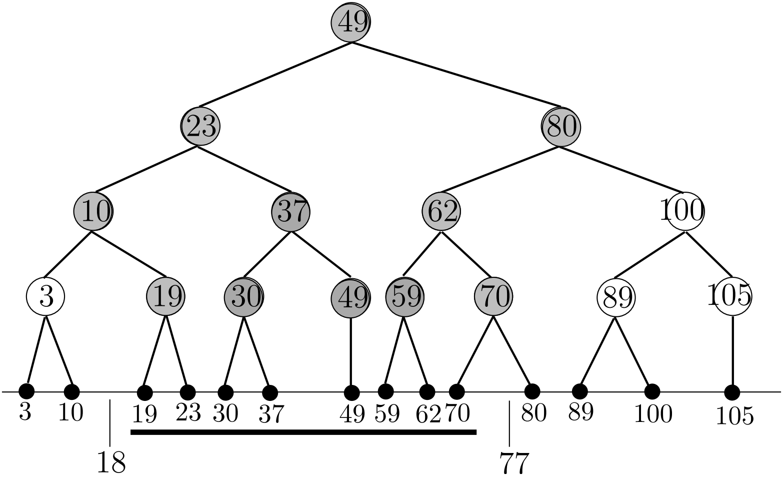

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- But this query time can not be avoided if we really have to report all the points.
- Actually, it can be seen that our algorithm is output-sensitive, meaning that its running time is sensitive to the size of the output.

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- To summarize, we have the result:

RESULT

Theorem

Let P be a set of n points in 1-dimensional space. The set P can be stored in a balanced binary search tree, which uses $O(n)$ storage and has $O(n \log n)$ construction time, such that the points in a query range can be reported in time $O(k + \log n)$, where k is the number of points reported.

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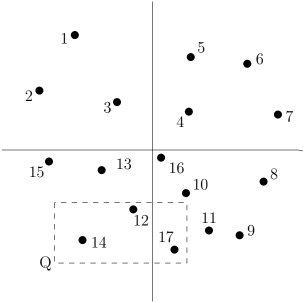
Given a set P of n points in the plane, report points inside a query rectangle $Q := [x : x'] \times [y : y']$ *whose sides are parallel to the axes.*



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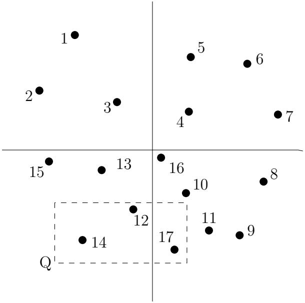

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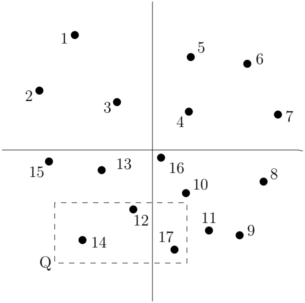
- For the example shown in the Figure, the answer is 14, 12, and 17.
- We assume that no two points in P have the same x-coordinate and no two points have the same y-coordinate.



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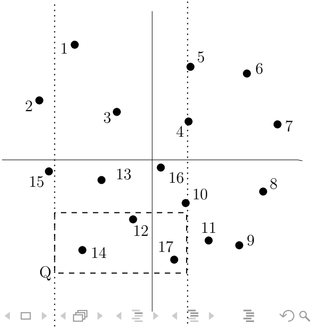

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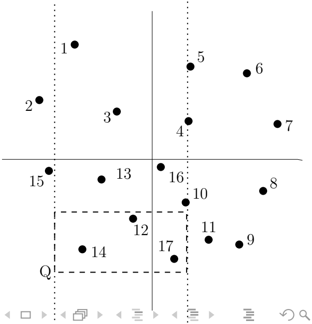

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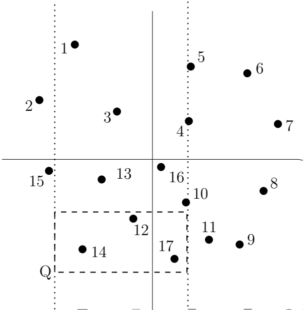
- We, however, can try splitting both x- and y-coordinates alternatively. This leads to a data structure called Kd-tree.



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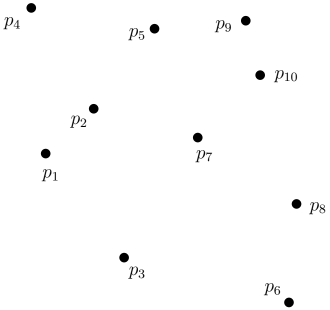
  - At the grand children of the root, we split again by a vertical line.
  - We split vertically at even depths and split horizontally at odd depths.

EXAMPLE



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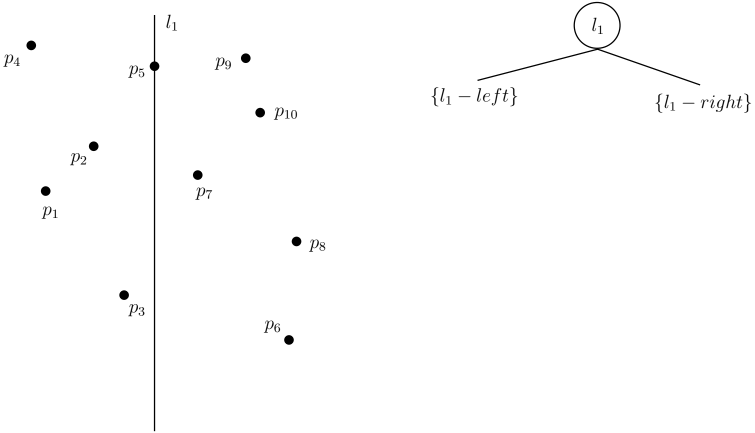

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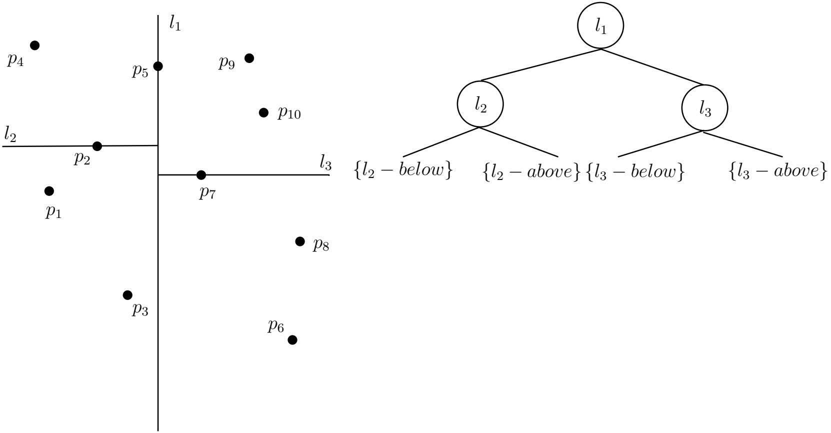
l_1
l_2
l_3
p_1
p_2
p_3
p_4
p_5
p_6
p_7
p_8
p_9
p_{10}
$\{l_2 - below\}$
$\{l_2 - above\}$
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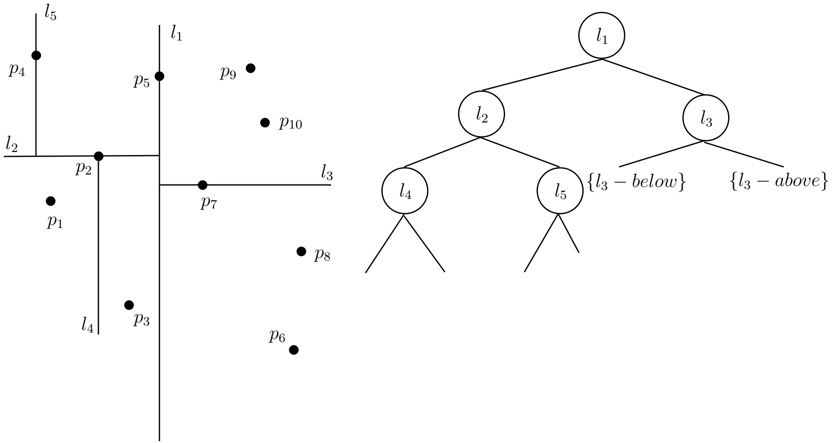
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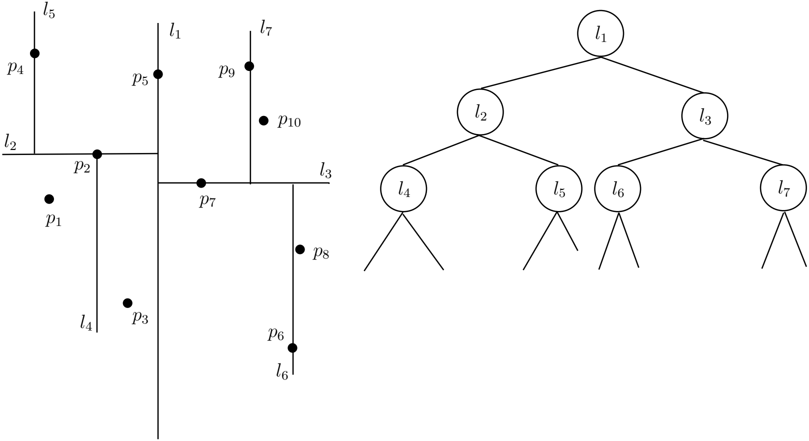

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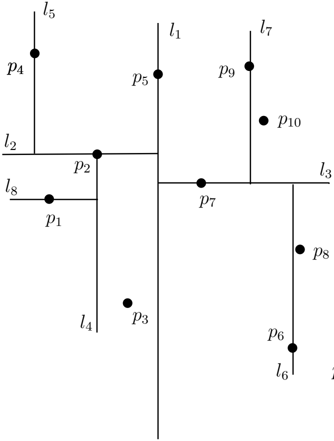

l_5
l_1
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p_{10}
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p_2
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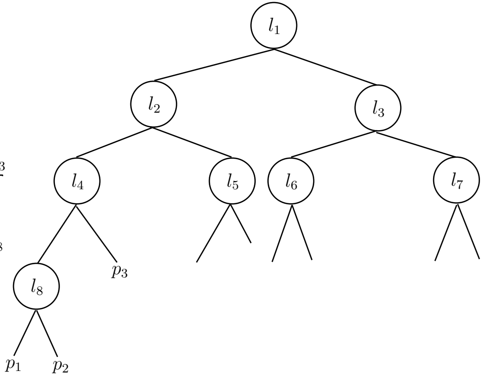
p_8
p_3
l_4
p_6
l_6



l_1
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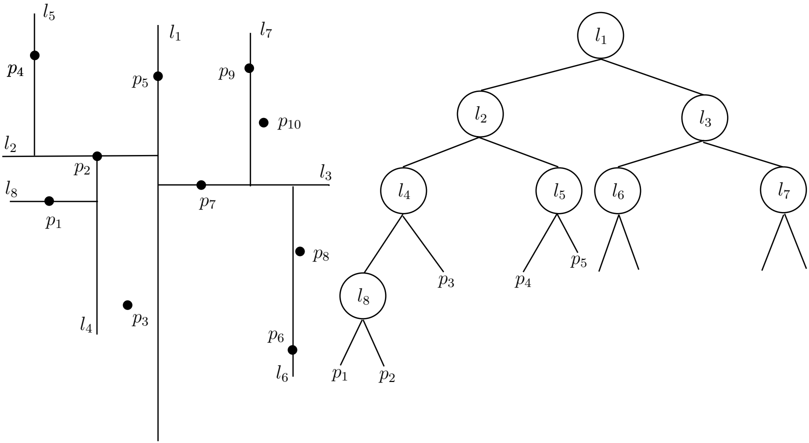
EXAMPLE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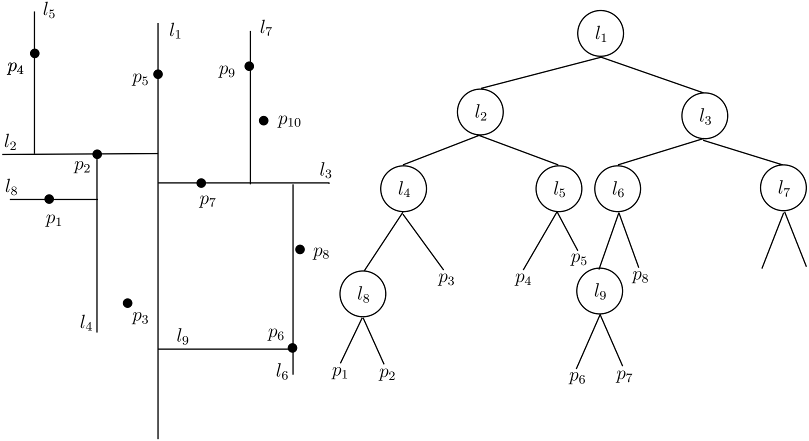
EXAMPLE



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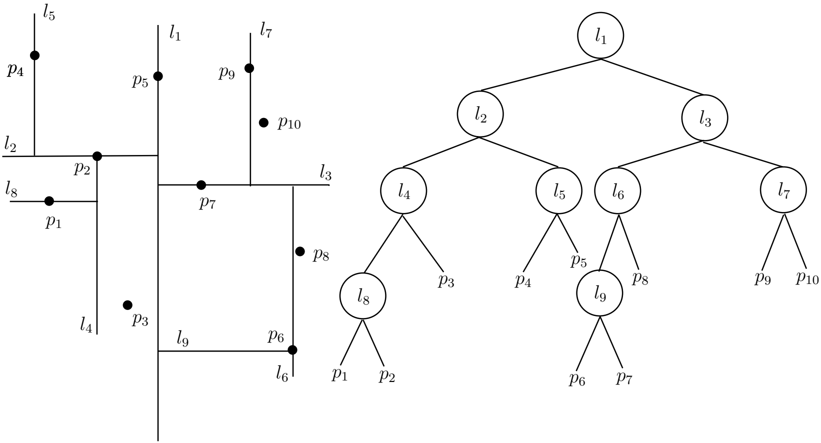
EXAMPLE



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p_8
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CONSTRUCTION

```
Algorithm BuildKdTree(P, depth)
  If P contains only one point
    then return a leaf storing this point.
  else if depth is even
    l := vertical line through median
           x-coordinates of the points in P
    P1 := set of points to the left of or on l
    P2 := set of points to the right of l
  else
    l := horizontal line through median
           y-coordinates of the points in P
    P1 := set of points below or on l
    P2 := set of points above l
  v-left = BuildKdTree(P1, depth+1)
  v-right = BuildKdTree(P2, depth+1)
  return a node v with value l, left child
           v-left, and right child v-right
```

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- In the pseudocode, median of a set of n numbers is assumed to be the $\lfloor n/2 \rfloor$-th smallest number.

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$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(\lceil n/2 \rceil) + O(n) & \text{if } n > 1 \end{cases}$$

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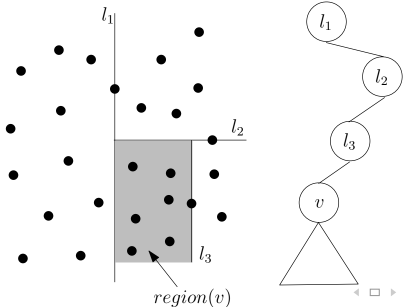
- Following figure makes this clear:



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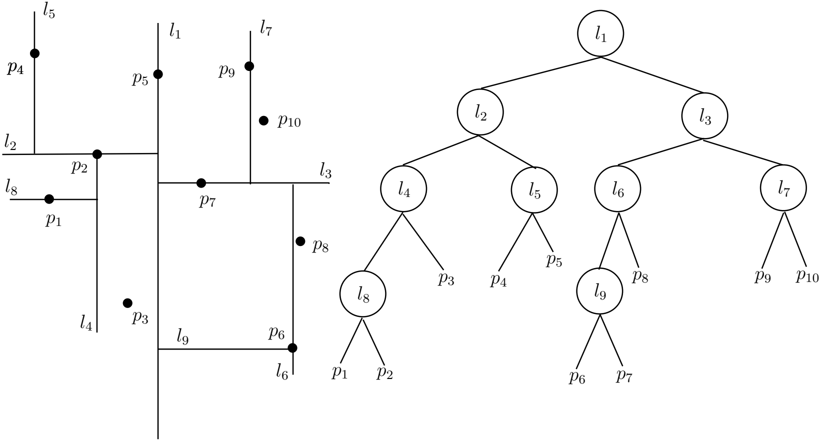
- An example follows.

EXAMPLE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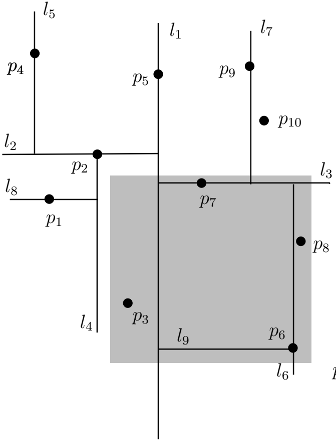
EXAMPLE



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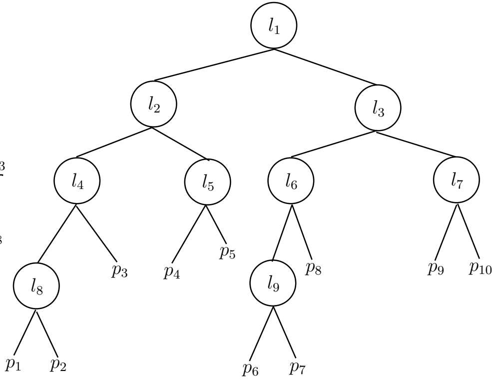

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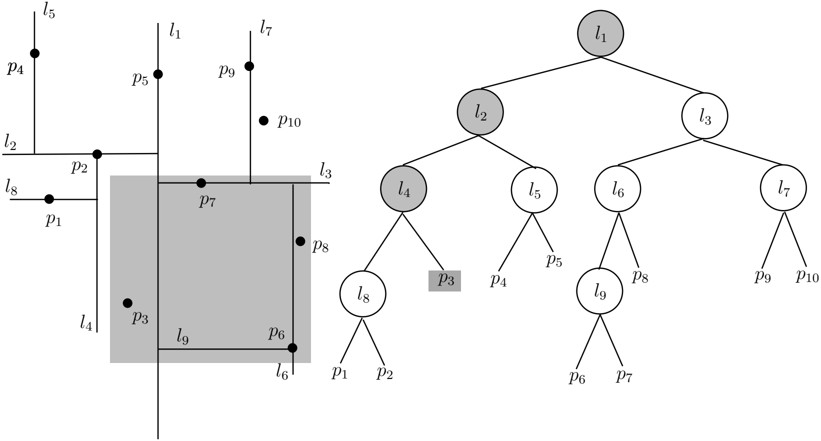
EXAMPLE



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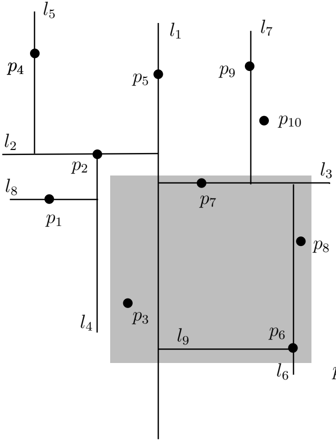

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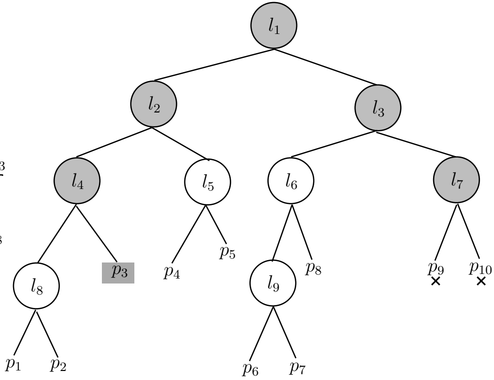

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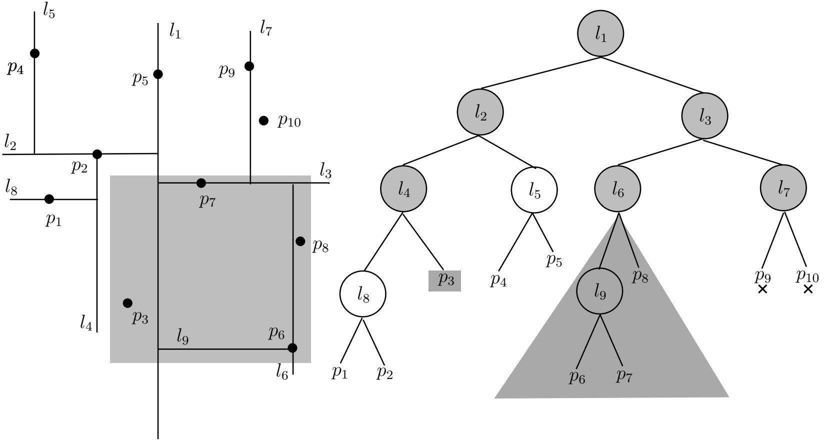
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PSEUDOCODE

```
Algorithm SearchKdTree(v, Q)
  If v is a leaf
    then report the point stored at v if it
           lies in Q
  else if region(lc(v)) is fully contained in Q
    then ReportSubtree(lc(v))
  else if region(lc(v)) intersects Q
    then SearchKdTree(lc(v), Q)
    If region(rc(v)) is fully contained in Q
      then ReportSubtree(rc(v))
    else if region(rc(v)) intersects Q
      then SearchKdTree(rc(v), Q)
```

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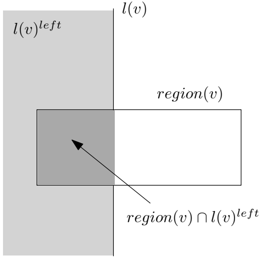
- Compute $region(v)$ for all nodes v during the preprocessing phase and store it.
- Maintain the current region through the recursive calls using the lines stored in the internal nodes.
  For instance, the region corresponding to the left child of a node v at even depth can be computed as
  $region(lc(v)) = region(v) \cap l(v)^{left}$
  where $l(v)$ is the splitting line stored at v, and $l(v)^{left}$ is the half-plane to the left of and including $l(v)$.



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- An upper bound for the number of such nodes is given by the recurrence relation

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- Solution of this recurrence is $O(\sqrt{n})$.

RESULT

Theorem

A kd-tree for a set P of n points in the plane uses $O(n)$ space and can be built in $O(n \log n)$ time. A rectangular range query on the kd-tree takes $O(\sqrt{n} + k)$ time, where k is the number of reported points.

Range tree

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- We have observed earlier that a 2-dimensional range query is essentially composed of two 1-dimensional sub-queries, one on the x-coordinate of points and one on the y-coordinate.
- One idea from this observation leads to Kd-tree. We now use the same observation in a different way to obtain range tree.

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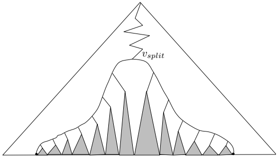

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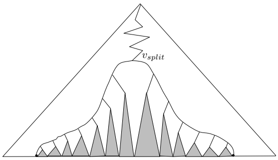
- Recall that our algorithm for this is to first locate the split node and then traverse along two search paths towards the leaves. While traversing the search paths, our algorithm also reported fully all the subtrees lying between the search paths.
- For any query range, $O(\log n)$ mutually disjoint subtrees are selected.



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- Hence, the subset of points whose x-coordinate lies in a query range can be expressed as the disjoint union of $O(\log n)$ canonical subsets. Of these points, those having y-coordinate in the interval $[y : y']$ are to be reported.
- This implies that, after performing 1-dimensional query on x-coordinate, we have to perform $O(\log n)$ 1-dimensional query on y-coordinate each on an appropriate $P(v)$.

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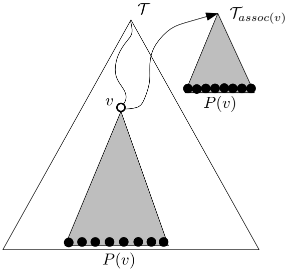
- The main tree is a balanced binary tree $\mathcal{T}$ built on the x-coordinate of the points in P.
- For any internal or leaf node v in $\mathcal{T}$, the canonical subset $P(v)$ is stored in a balanced binary search tree $\mathcal{T}_{assoc(v)}$ on the y-coordinate of the points. The node v stores a pointer to the root of $\mathcal{T}_{assoc(v)}$, which is called the associated structure of v.



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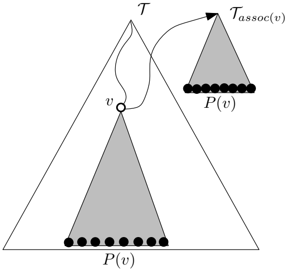

- This multi-level data structure is called range tree.

Construction

```
Algorithm Build2dRangeTree(P)
  Build the associated BBST Tassoc on the set Py
     of y-coordinates of the points in P. Leaves
     of Tassoc stores both the points and their
     y-coordinates.
  If P contain only one point
    then create a leaf v storing this point and
       make Tassoc the associate structure of v
  else
    Split P into two subsets Pleft containing
       points with x-coordinates less than or
       equal to median x-coordinate and Pright
       containing points with x-coordinates
       greater than median x-coordinate
```

CONSTRUCTION

```
        vleft := Build2dRangeTree(Pleft)
        vright := Build2dRangeTree(Pright)
        Create node v containing the median
           x-coordinate, vleft as left child, vright
           as right child, and Tassoc as associated
           structure
    return v
```

Complexity

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- Regarding time complexity, we assume that the points are maintained in two lists, one sorted on x-coordinate of the points and the other on y-coordinate. Then from similar arguments as used in storage complexity the time complexity follows.

QUERY

- The query algorithm first selects, by the application of our 1-dimensional query algorithm, $O(\log n)$ canonical subsets that together contain points whose x-coordinate lie in the range $[x : x']$.

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- Of these subsets, we then report the points whose y-coordinate lie in the range $[y : y']$. This can be done by applying the same 1-dimensional query algorithm on each of the $O(\log n)$ associated structures that stores the selected canonical subsets.

Complexity

- At each node v in the main tree $\mathcal{T}$ we spend constant time to decide whether to go left or right and possibly call 1-dimensional query algorithm on the associated structure. Total time spent for v is thus $O(\log n + k_v)$, where k_v is the number of points reported.

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- Hence the total time spent on a search path is $\sum(O(\log n + k_v)$, where the summation extends over the nodes in the search path.
- Since there are $O(\log n)$ nodes in a search path and two such paths are involved in each query, total time is $O(\log^2 n + k)$ where $k = \sum k_v$ is the total number of points reported.

Result

Theorem

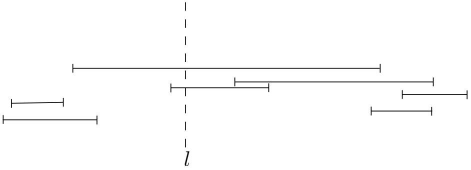
Let P be a set of n points in the plane. A range tree for P uses $O(n \log n)$ storage and can be constructed in $O(n \log n)$ time. By querying this range tree one can report the points in P that lie in a rectangular query range in $O(\log^2 n + k)$ time where k is the number of reported points.

OUTLINE

SEGMENT SEARCHING PROBLEM

Problem

Given a set S of n horizontal line segments in the plane, preprocess them such that the segments intersecting a vertical query line l can be reported efficiently.



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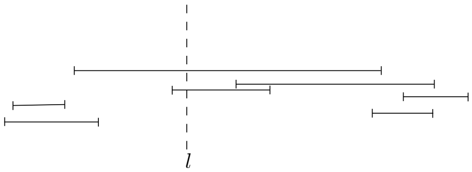

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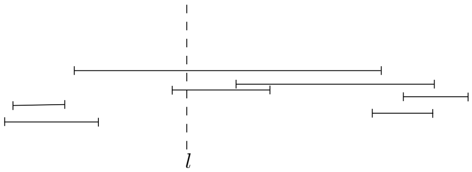

- Obviously, the problem can be solved in $O(n)$ time and there exists instances for which we have to report all the segments.
- However, we are interested in an output-sensitive algorithm which is efficient in multi-shot query scenario.

Interval searching problem

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- Hence the problem becomes 1-dimensional and can be stated as follows.

Problem

Given a set of intervals on the real line, report the ones that contain the query point q_x.

DATA STRUCTURE

- Let $I = \{[x_1 : x'_1], [x_2 : x'_2], \ldots, [x_n : x'_n]\}$ be the set of n closed intervals on the real line.

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- We construct a binary search tree based on this idea.

Data structure

- The root of the tree contains x_{mid}. The right subtree of the tree stores the set I_{right} of the intervals lying completely to the right of x_{mid}, and the left subtree stores the set I_{left} of intervals completely to the right of x_{mid}.

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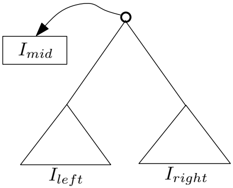
- The set I_{mid} of intervals containing x_{mid} is stored in a separate structure and we associate that structure with the root of our tree.



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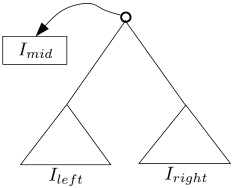

- The subtrees are constructed recursively in the same way.

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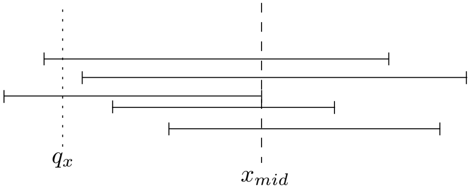

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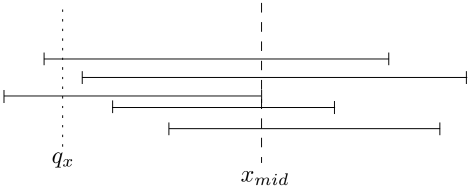

- We thus maintain two sorted lists of the intervals in I_{mid}, one sorted on left endpoints and the on right endpoints. A traversal of the appropriate list enable us to report intervals containing q_x in time proportional to the number of intervals reported.

INTERVAL TREE

- The whole data structure we thus arrived at for storing a given set I of intervals is called an interval tree.

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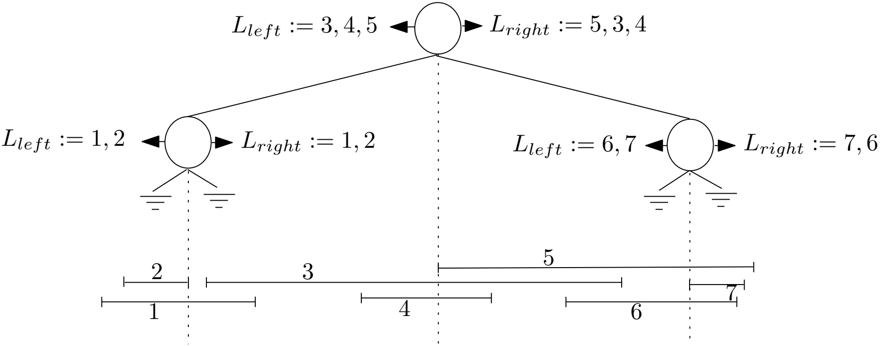
- Following figure shows an example.



CONSTRUCTION

```
Algorithm ConstructIntervalTree(I)
  If I = Null
    then return an empty leaf
  Create a node v. Compute x-mid, the median
     of the set of interval endpoints and
     store x-mid with v
  Compute I-mid and construct two sorted lists
     for I-mid: a list L-left(v) sorted on left
     endpoint and a list L-right(v) sorted on
     right endpoint
  Store these two lists at v
  lc(v) := ConstructIntervalTree(I-left)
  rc(v) := ConstructIntervalTree(I-right)
  return v
```

COMPLEXITY

- Observe that each interval is only stored in a set I_{mid} once and, hence, only appears once in each of the two sorted lists. So total amount of storage required for all associated lists is bounded by $O(n)$. The tree itself uses $O(n)$ storage. Hence an interval tree on n intervals requires $O(n)$ storage.

Complexity

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- An interval tree on n intervals has $O(\log n)$ depth.
- We assume that the set of endpoints are presorted. So median can be computed in constant time. I_{mid}, I_{left}, and I_{right} can be computed in $O(n)$ time. Creating the lists $L_{left}(v)$ $L_{right}(v)$ takes $O(|I_{mid}| \log |I_{mid}|)$ time. Hence time spend for creating the node v (not counting the recursive calls) is $O(n + |I_{mid}| \log |I_{mid}|)$. So total construction time is $O(n \log n)$.

QUERY PSEUDOCODE

```
Algorithm QueryIntervalTree(v,qx)
  If v is not a leaf
    then if qx < x-mid(v)
           then Walk along the list L-left(v),
                   starting at the interval with
                   the leftmost endpoint, reporting
                   all the intervals that contain
                   qx.
              QueryIntervalTree(lc(v),qx)
         else
           Walk along the list L-right(v),
                   starting at the interval with
                   the rightmost endpoint, reporting
                   all the intervals that contain
                   qx.
              QueryIntervalTree(rc(v),qx)
```

Query complexity

- At any node v we visit, we spend $O(1 + k_v)$ time, where k_v is the number of intervals that we report at v.

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- The sum of the k_v's over all visited nodes is k, total number of intervals reported.
- Depth of the tree is $O(\log n)$.
- Hence, total query time is $O(\log n + k)$.

RESULT

Theorem

An interval tree for a set I of n intervals uses $O(n)$ *storage and can be built in* $O(n \log n)$ *time. Using the interval tree we can report all intervals that contain a query point in* $O(\log n + k)$ *time, where k is the number of reported intervals.*

SEGMENT SEARCH WITH VERTICAL QUERY SEGMENT

- Let us consider a slightly more difficult segment search problem.

Problem

Given a set S of n horizontal line segments in the plane, preprocess them such that the segments intersecting a vertical query segment q can be reported efficiently.

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- Let us consider a slightly more difficult segment search problem.

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Given a set S of n horizontal line segments in the plane, preprocess them such that the segments intersecting a vertical query segment q can be reported efficiently.

- We next show that the problem can be solved by using a data structure which is a modified form of interval tree.

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- If we use our `QueryIntervalTree` procedure to query $\mathcal{T}$ with a vertical query segment q, we can traverse the tree properly but problem arise when trying to report segments from I_{mid} containing q.

SEGMENT SEARCH WITH VERTICAL QUERY SEGMENT

- Suppose q_x lies to the left of x_{mid}. For a segment $s \in I_{mid}$ to be intersected by q, it is not sufficient that its left endpoint lies to the left of q, it is also required that its y-coordinate lies in the range $[q_y : q'_y]$.

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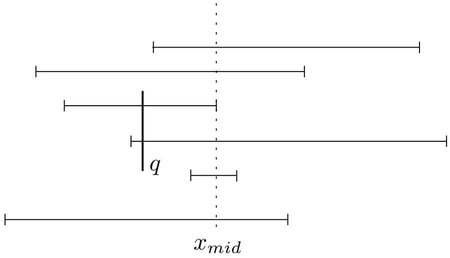

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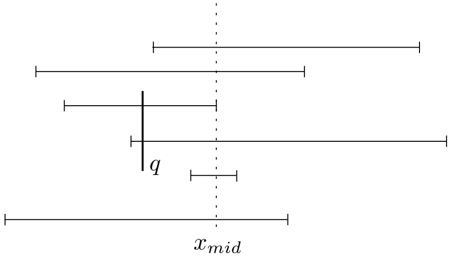
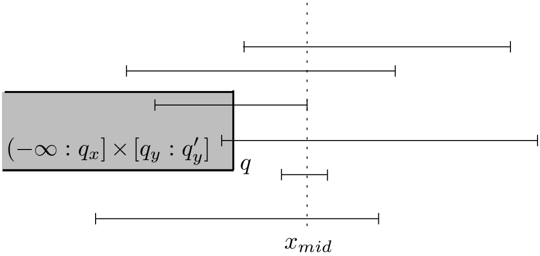


- So storing the endpoints in an ordered list is not enough.

SEGMENT SEARCH WITH VERTICAL QUERY SEGMENT

- We need an associated structure such that, given a query range $(-\infty : q_x] \times [q_y : q'_y]$, we must be able to report all the segments whose left endpoints lies in that range.



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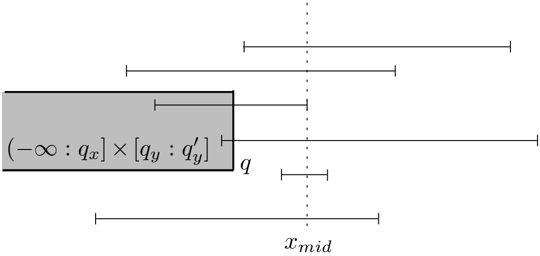

- Similarly, if q lies to the right of x_{mid}, we must be able to report all the segments whose right endpoints lies in the range $[q_x : +\infty) \times [q_y : q'_y]$.

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- The main structure is an interval tree $\mathcal{T}$ on the x-intervals of the segments.
- Instead of the sorted lists $L_{left}(v)$ and $L_{right}(v)$, we have two range trees: a range tree $\mathcal{T}_{left}(v)$ on the left endpoints of the segments in $I_{mid}(v)$, and a range tree $\mathcal{T}_{right}(v)$ on the right endpoints of the segments in $I_{mid}(v)$.

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Segment search with vertical query segment

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- The total query time therefore becomes $O(\log^2 n + k)$.

RESULT

Theorem

Let S be a set of n horizontal segments in the plane. The segments intersecting a vertical query segment can be reported in $O(\log^2 n + k)$ time with a data structure that uses $O(n \log n)$ storage, where k is the number of reported segments. The structure can be built in $O(n \log n)$ time.

WINDOWING QUERY PROBLEM

- The tools we have thus developed can be used to solve another important query problem, known as windowing query problem. A simplified version of the problem is as follows.

Problem

Let S be a set of n axis-parallel and mutually disjoint line segments in the plane. Preprocess the segments such that segments intersecting a query window $W := [x : x'] \times [y : y']$ can be reported efficiently.

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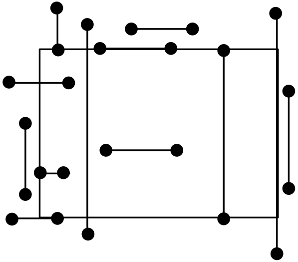

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- Remaining intersecting segments cross window boundary twice.
- First category of intersecting segments can be identified by using the range query data structure we have developed.
- The segments that intersect window boundary twice can be identified by the application of our modified interval tree data structure twice: for determining horizontal segments that intersect one of the two vertical edges of the window and for determining vertical segments that intersect one of the two horizontal edges of the window (by reversing the role of x- and y-coordinates this can be dealt with).

WINDOWING QUERY PROBLEM

- We now extend the windowing query problem to accommodate segments of arbitrary orientations.

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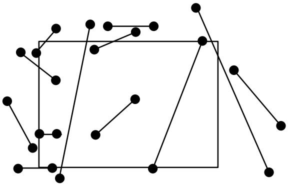

- Though segments which have at least one endpoint inside the window can be determined as before, interval tree can no longer be used to find segments which intersect the window twice.

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- It can be seen that, for solving the windowing query problem, it is sufficient to apply the procedure taking each of the four boundary edges of the window as the query segment.

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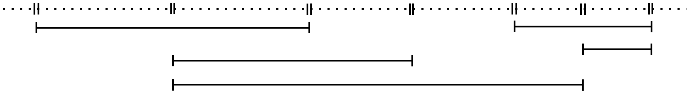

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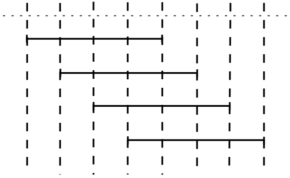
- However, observe that in this approach intervals that span a lot of elementary intervals are stored at many leaves increasing thereby the storage required. In the worst case, amount of storage may become quadratic.



SEGMENT TREE

- Build a binary search tree $\mathcal{T}$ whose leaves corresponds to these elementary intervals. We denote the elementary interval corresponding to a leaf μ by $Int(\mu)$.
- If all the intervals in I containing $Int(\mu)$ would be stored at the leaf μ, then we would report the k intervals containing q_x in $O(\log n + k)$ time. So the query could be answered efficiently.

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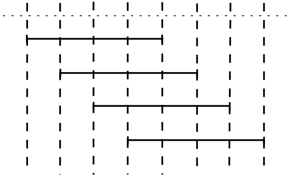
- For this reason, instead of storing at leaf level, intervals are stored as high as possible.



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  - Internal nodes of $\mathcal{T}$ correspond to intervals that are the union of the intervals of its two children, i.e., union of elementary intervals $Int(\mu)$ of the leaves in the subtree rooted by it.
  - Each internal node or leaf v in $\mathcal{T}$ stores the interval $Int(v)$ and a set $I(v) \subseteq I$ of intervals (e.g., in a linked list). This canonical subset of node v contains the intervals $[x : x'] \in I$ such that $Int(v) \subseteq [x : x']$ and $Int(parent(v)) \not\subseteq [x : x']$.

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- It remains to compute the canonical subsets for the nodes. For this, we insert the intervals one by one into the segment tree. Code for insertion is as follows.

CONSTRUCTION

```
Algorithm InsertSegmentTree(v,[x:x′])
  If Int(v) is a subset of [x:x′]
    then store [x:x′] at v
  else if Int(lc(v)) and [x:x′] are not disjoint
    then InsertSegmentTree(lc(v),[x:x′])
    If Int(rc(v)) and [x:x′] are not disjoint
    then InsertSegmentTree(rc(v),[x:x′])
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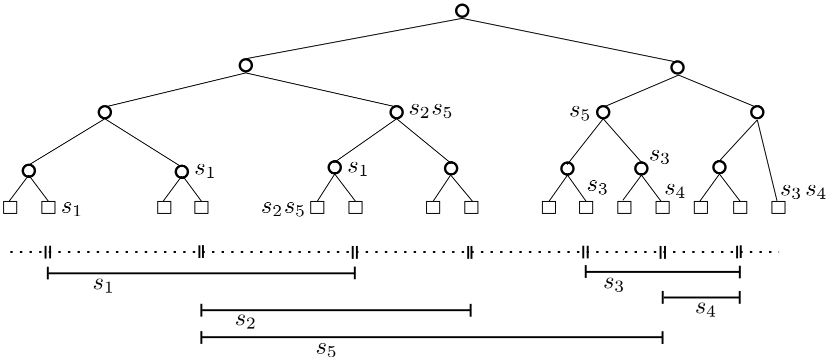
- Total time required for constructing a segment tree is $O(n \log n)$. Space requirement is also $O(n \log n)$ (Prove!).

EXAMPLE



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- Query algorithm is simple:

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Algorithm QuerySegmentTree(v,qx)
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- The query algorithm visits one node per level, so $O(\log n)$ nodes in total. At a node v we spend $O(1 + k_v)$ time, where k_v is the number of reported intervals. Hence query time is $O(\log n + k)$, where k is total number of reported intervals.

RESULT

Theorem

A segment tree for a set I of n intervals uses $O(n \log n)$ storge and can be constructed in $O(n \log n)$ time. Using the segment tree we can report all intervals that contain a query point in $O(\log n + k)$ time, where k is the number of reported segments.

WINDOWING QUERY PROBLEM

- We now go back to our windowing query problem. The problem we wanted to solve is:

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Let S be a set of n mutually disjoint line segments with arbitrary orientations in the plane. Preprocess the segments such that segments intersecting a vertical query segment $q := q_x \times [q_y : q'_y]$ *can be reported efficiently.*

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- We now show that segment tree data structure we have developed can be augmented to solve this problem.

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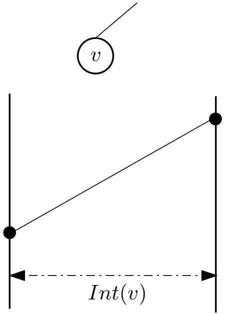

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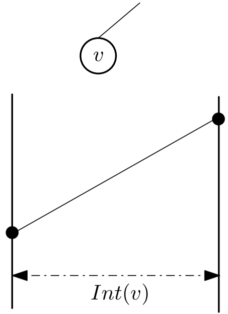

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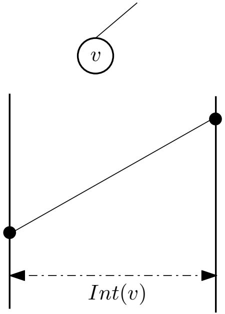
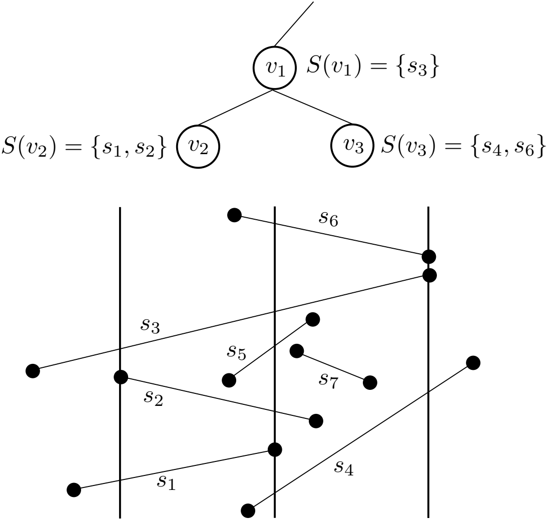


- Let us see an example.

EXAMPLE



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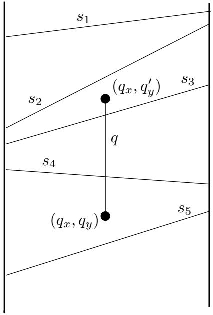

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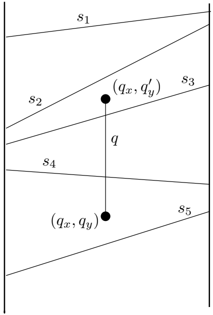
- A segment s in such a canonical subset is intersected by q iff the lower endpoint of q is below s and the upper endpoint is above s.
- To identify proper segments we use the fact that the segments in the canonical subset $S(v)$ span the slab corresponding to v and that they do not intersect each other – i.e., they can be vertically ordered.



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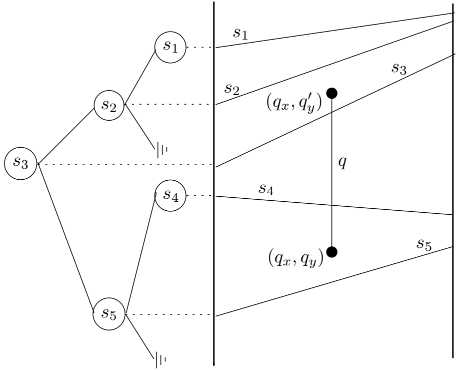
- Hence we can store $S(v)$ in a search tree $\mathcal{T}(v)$ according to their vertical order.



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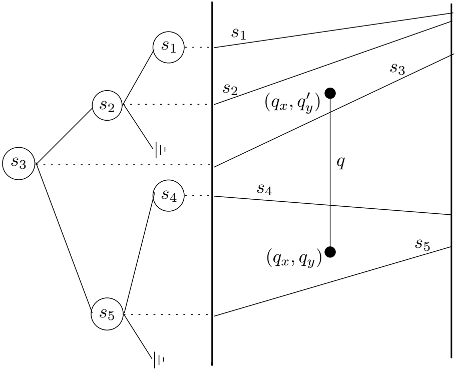
- Hence we can store $S(v)$ in a search tree $\mathcal{T}(v)$ according to their vertical order.
- By searching $\mathcal{T}(v)$ we can find the intersected segments in $O(\log n + k_v)$ time, where k_v is the number of intersected segments.



WINDOWING QUERY PROBLEM

- So the over all query algorithm is like this. We search with q_x in the segment tree in the usual way, and at every node v on the search path we search with upper and lower endpoint of q in $\mathcal{T}(v)$ to report the segments in $S(v)$. Since search in $\mathcal{T}(v)$ takes $O(\log n + k_v)$ time where k_v is the number of reported segments at v, total query time is $O(\log^2 n + k)$, where k is the total number of reported segments.

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- Because the associated structure of any node v uses storage linear in size of $S(v)$, total amount of storage remain $O(n \log n)$.
- The associated structure can be built in $O(n \log n)$ time, leading to a preprocessing time of $O(n \log^2 n)$.

Result

Theorem

Let S be a set of n disjoint segment of arbitrary orientations in the plane. The segments intersecting a vertical query segment can be reported in $O(\log^2 n + k)$ *time with a data structure that used* $O(n \log n)$ *storage, where k is the number of reported segments. The data structure can be built in* $O(n \log^2 n)$ *time.*

RESULT

Theorem

Let S be a set of n disjoint segment of arbitrary orientations in the plane. The segments intersecting an axis parallel rectangular query window can be reported in $O(\log^2 n + k)$ *time with a data structure that used* $O(n \log n)$ *storage, where k is the number of reported segments. The data structure can be built in* $O(n \log^2 n)$ *time.*

Outline

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- We have confined ourselves in the plane only. Each of the problems can be generalized in space and higher dimensions.

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- The queries we have considered are called reporting queries. Another type of queries is often important where we want to count instead of report. Such queries are called counting queries.
- Both objects to be searched and query shape can vary. For example they can be triangles, circles, ellipses, tetrahedrons, simplexes, in higher space and higher dimensions.

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Thank you!