Circles and Spheres

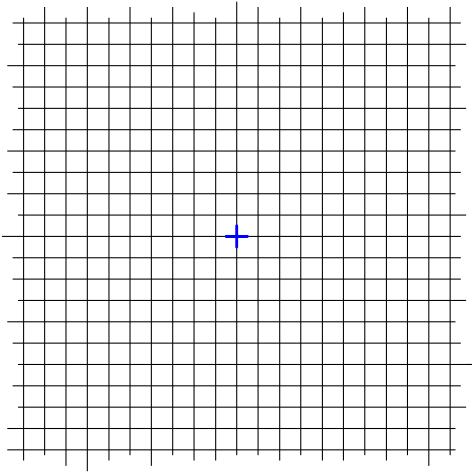
Anomalies and Algorithms in Digital Space

Partha Bhowmick

Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur
India

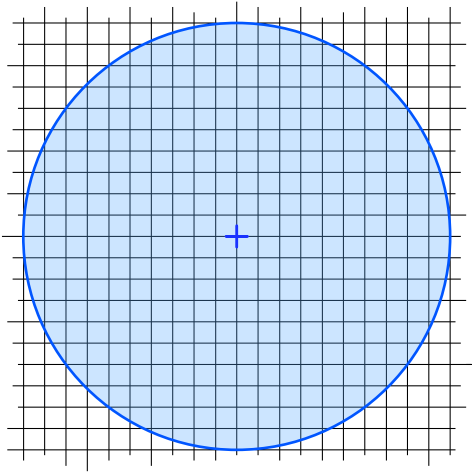
NIT Warangal **23 Oct 2013**

Circle in $\mathbb{Z}^2$



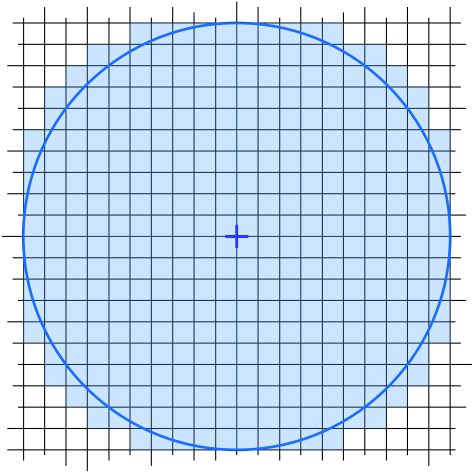
We assume: center $= (0, 0)$ and r is an integer.

Circle in $\mathbb{Z}^2$



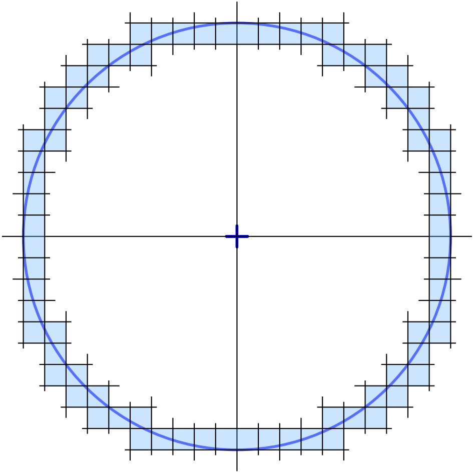
$r = 10$

Circle in $\mathbb{Z}^2$



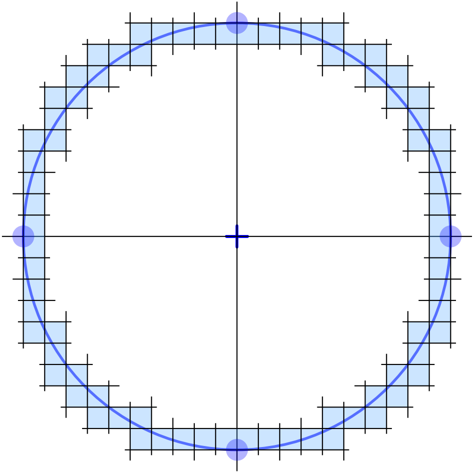
$r = 10$

Circle in $\mathbb{Z}^2$



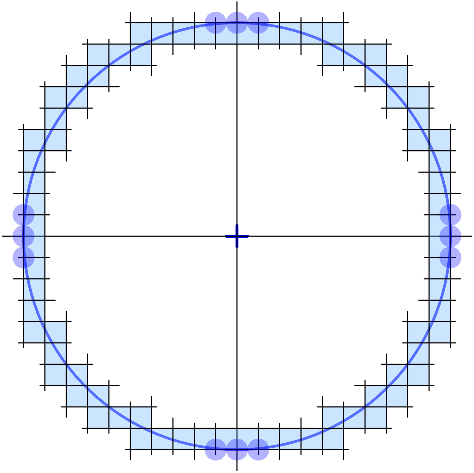
$r = 10$

Circle in $\mathbb{Z}^2$

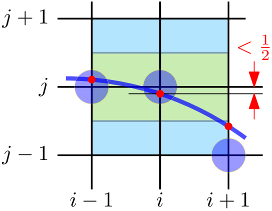


$r = 10$

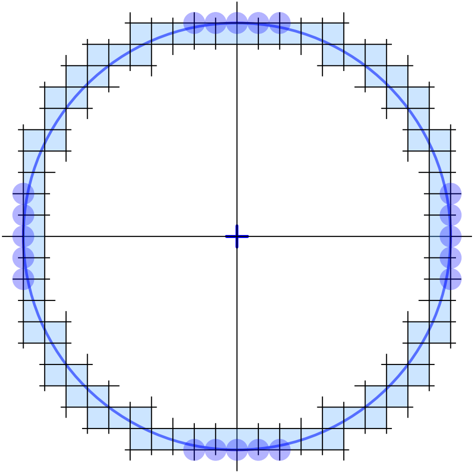
Circle in $\mathbb{Z}^2$



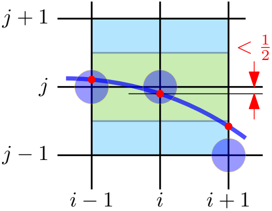
$r = 10$



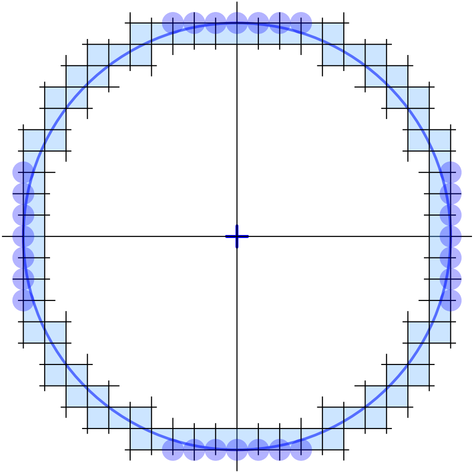
Circle in $\mathbb{Z}^2$



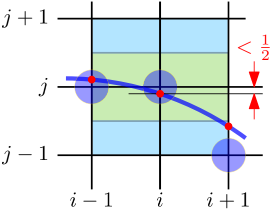
$r = 10$



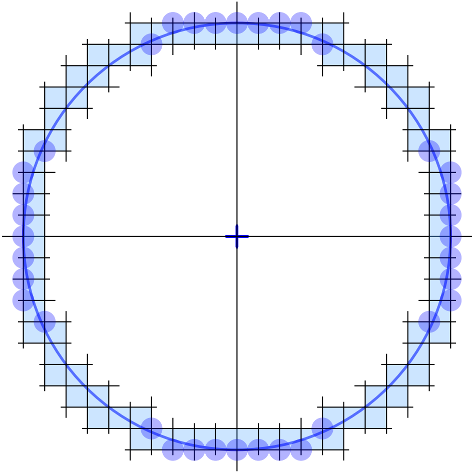
Circle in $\mathbb{Z}^2$



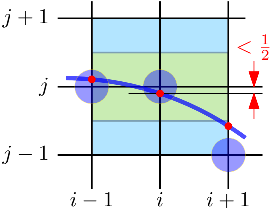
$r = 10$

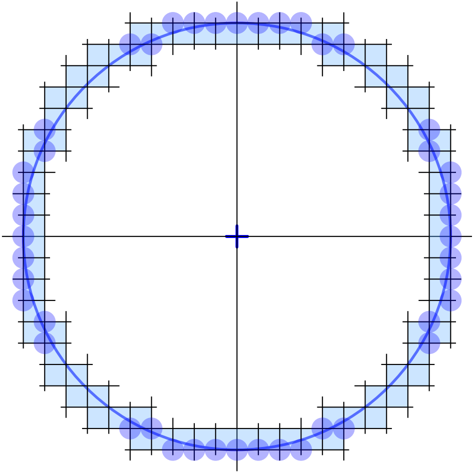


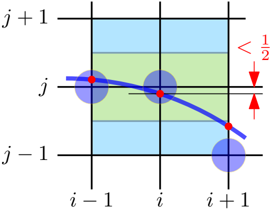
Circle in $\mathbb{Z}^2$



$r = 10$

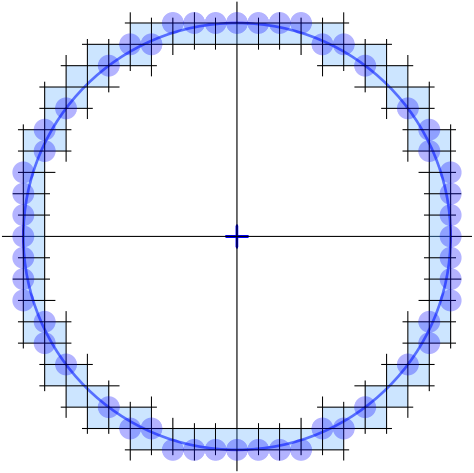


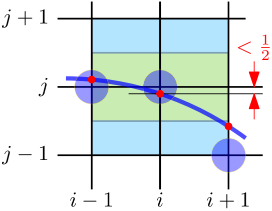




$r = 10$

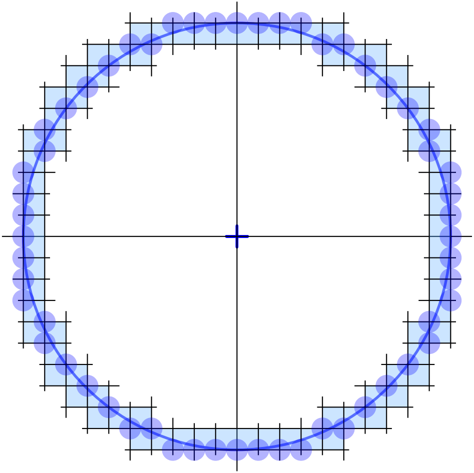
Circle in $\mathbb{Z}^2$



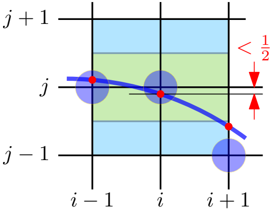


$r = 10$

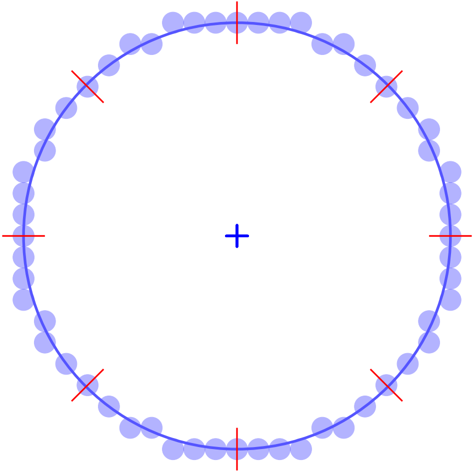
Circle in $\mathbb{Z}^2$



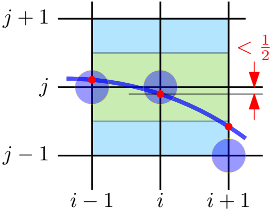
$r = 10$



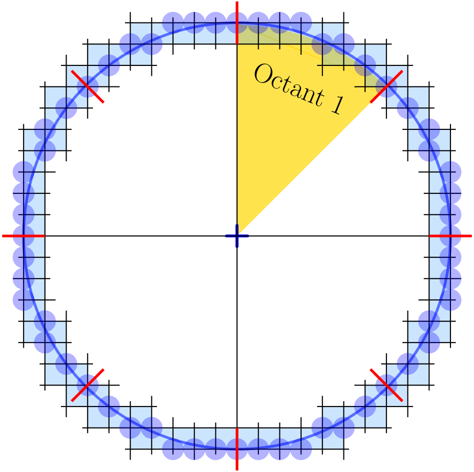
Circle in $\mathbb{Z}^2$

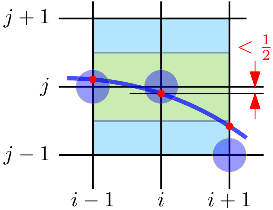


$r = 10$

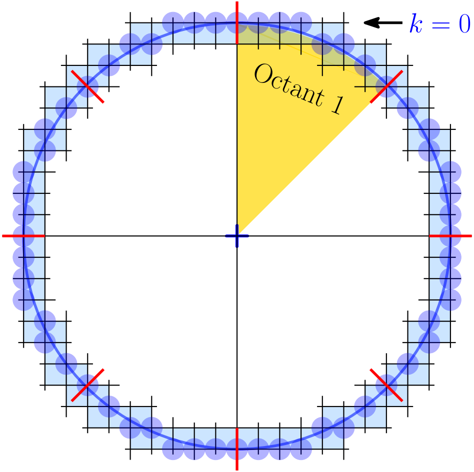


Circle in $\mathbb{Z}^2$



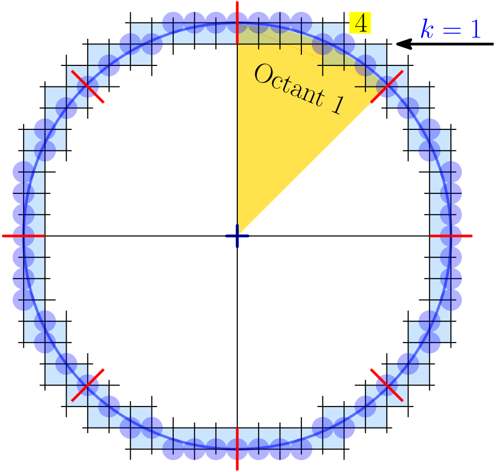


Circle in $\mathbb{Z}^2$



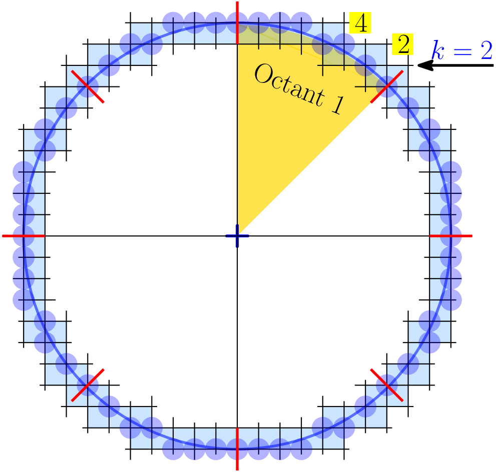
$\leftarrow k = 0 - s[0, r-1] = s[0, 9] = 4$

Circle in $\mathbb{Z}^2$



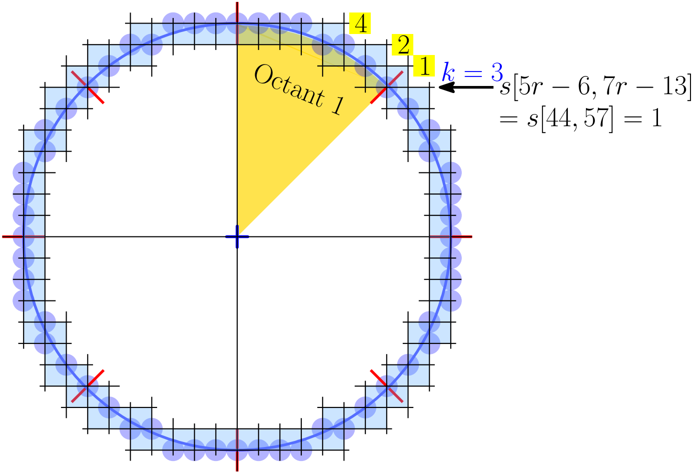
$s[r, 3r-3] = s[10, 27]$
$= 2$

Circle in $\mathbb{Z}^2$

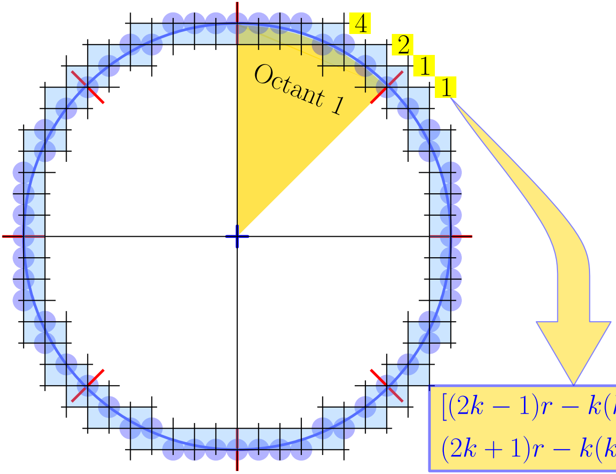


$s[3r-2, 5r-7]$
$= s[28, 43] = 1$

Circle in $\mathbb{Z}^2$



Circle in $\mathbb{Z}^2$



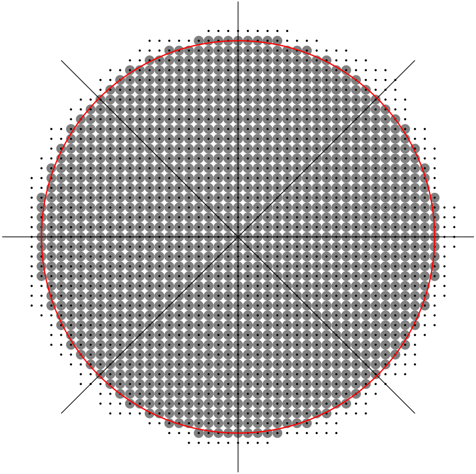
$$[(2k-1)r - k(k-1),$$
$$(2k+1)r - k(k+1) - 1]$$

Circle in $\mathbb{Z}^2$

Algorithm 1.1: DCS(r)

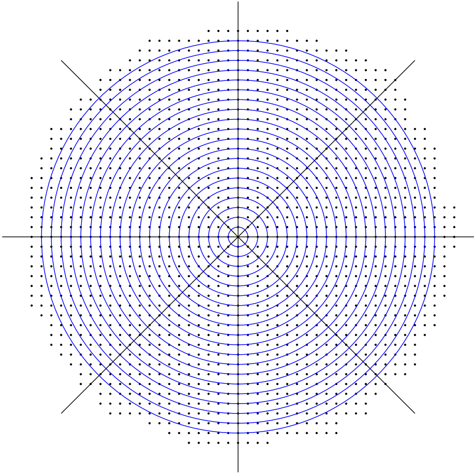
`int` $i \leftarrow 0, j \leftarrow r, s \leftarrow 0, w \leftarrow r - 1$
`int` $l \leftarrow 2w$
while $j \geqslant i$ **do**
    repeat
        select (i, j)
        $s \leftarrow s + 2i + 1$
    until $s > w$
    $w \leftarrow w + l$
    $l \leftarrow l - 2, j \leftarrow j - 1$

Disc in $\mathbb{Z}^2$



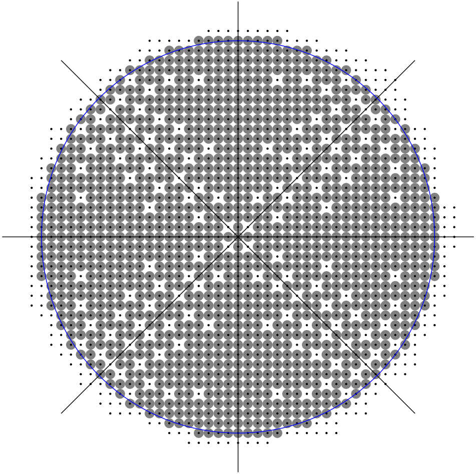
Quest: Can $\mathcal{D}^{\mathbb{Z}}(r)$ be covered by circle lattice points?

Disc in $\mathbb{Z}^2$



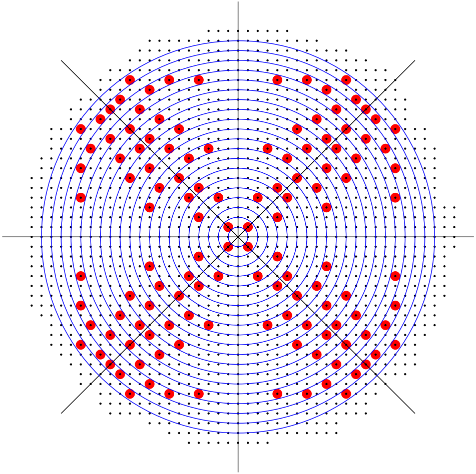
$$\left\{\mathcal{C}^{\mathbb{R}}(s) : 0 \leqslant s \leqslant r = 20\right\}$$

Disc in $\mathbb{Z}^2$



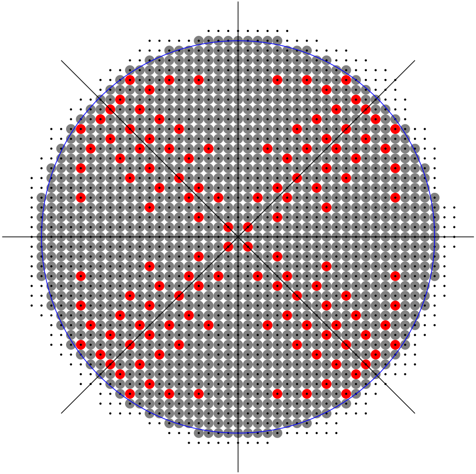
$$\bigcup_{s=0}^{r} \mathcal{C}^{\mathbb{Z}}(s)$$

Disc in $\mathbb{Z}^2$



$$\mathcal{A}^{\mathbb{Z}}(r) = \mathcal{D}^{\mathbb{Z}}(r) \setminus \bigcup_{s=0}^{r} \mathcal{C}^{\mathbb{Z}}(s)$$

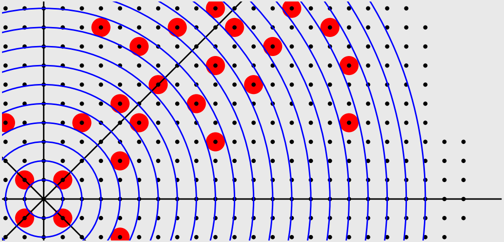
Disc in $\mathbb{Z}^2$



$\mathcal{D}^{\mathbb{Z}}(r)$

Disc in $\mathbb{Z}^2$

Issues



- Order of absentee count—any idea ?!
- Absentee characterization
- Algorithm for fixing them

Anomalies in $\mathbb{Z}^2$ (1)

Theorem

[a] *The squares of abscissae of the pixels in* $\mathcal{C}_1^{\mathbb{Z}}(r)$ *whose ordinates are* j *lie in the interval* $I_{r-j}^{(r)} = \left[u_{r-j}^{(r)}, v_{r-j}^{(r)}\right)$, *where*

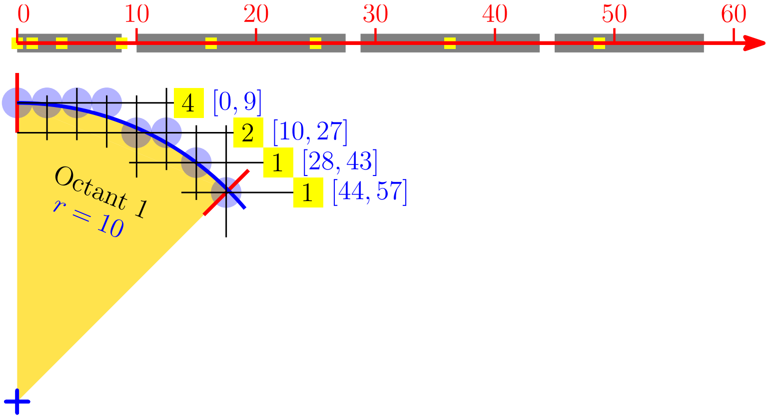
$$u_{r-j}^{(r)} = r^2 - j^2 - j,$$

$$v_{r-j}^{(r)} = r^2 - j^2 + j.$$

[a] P. Bhowmick and B. B. Bhattacharya, Number-theoretic interpretation and construction of a digital circle, ***Discrete Applied Mathematics***, **156** : 2381–2399, **2008.**

Anomalies in $\mathbb{Z}^2$

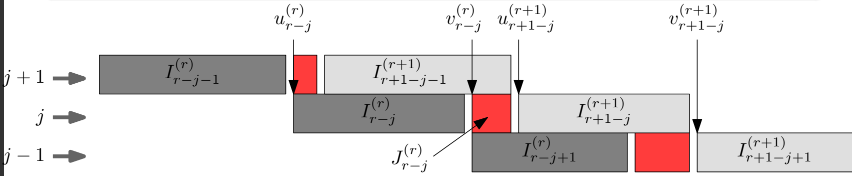
$$\left[u_{r-j}^{(r)} = \max(0, r^2 - j^2 - j),\ v_{r-j}^{(r)} = r^2 - j^2 + j\right)$$



Anomalies in $\mathbb{Z}^2$ (3)

Lemma

For $r > 0$, the intervals $I^{(r)}_{r-j}$ and $I^{(r+1)}_{r+1-j}$ are disjoint, with $u^{(r+1)}_{r+1-j} > v^{(r)}_{r-j}$.

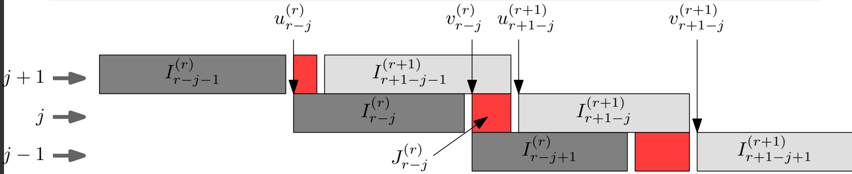


Anomalous intervals (red)

Anomalies in $\mathbb{Z}^2$ (4)

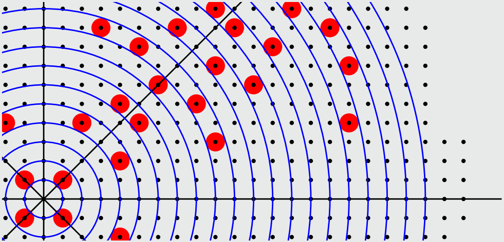
Lemma

(i, j) is an absentee if and only if i^2 lies in $J^{(r)}_{r-j} := \left[v^{(r)}_{r-j}, u^{(r+1)}_{r+1-j}\right)$ for some $r \in \mathbb{Z}^+$.



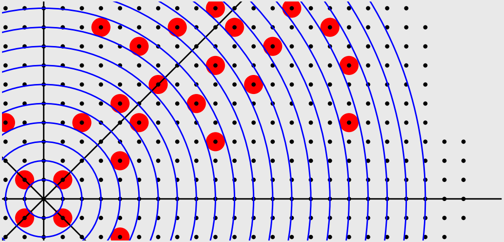
Anomalies in $\mathbb{Z}^2$

Example



(2, 4): For $r = 4$, $v_{r-j}^{(r)} = r^2 - j^2 + j = 16 - 16 + 4 = 4$,
$u_{r+1-j}^{(r+1)} = (r+1)^2 - j^2 - j = 25 - 16 - 4 = 5$
$\Rightarrow J_0^{(4)} = [4, 5) = [4, 4]$ which contains 2^2
$\Rightarrow$ (2, 4) is an absentee.

Theorem

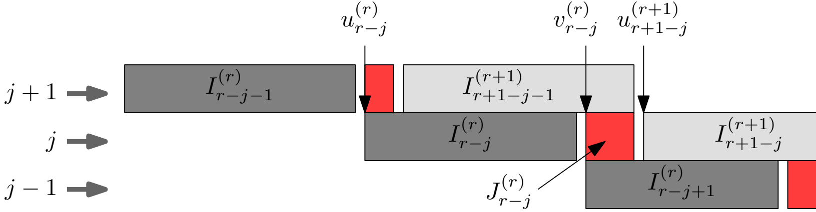


[a] *A pixel $p(i,j)$ is an absentee if and only if* $i^2 \in J^{(r)}_{r-j}$, *where* $r = \max\left\{s \in \mathbb{Z} : s^2 < i^2 + j^2\right\}$.

[a] S. Bera *et al.*, On Covering a Digital Disc with Concentric Circles in $\mathbb{Z}^2$, ***Theoretical Computer Science: 506***, 1–16, 2013.

Recap: $J^{(r)}_{r-j} := \left[v^{(r)}_{r-j}, u^{(r+1)}_{r+1-j}\right) = \left[u^{(r)}_{r-j} = \max(0, r^2 - j^2 - j),\ v^{(r)}_{r-j} = r^2 - j^2 + j\right)$

Anomalies in $\mathbb{Z}^2$

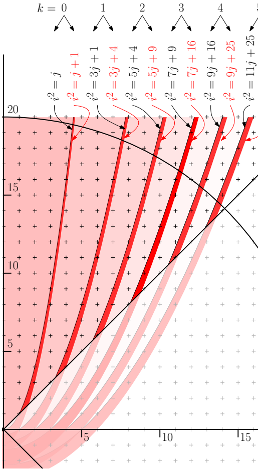


If $p(i,j)$ lies in $k(=r-j)$th run of $\mathcal{C}^{\mathbb{Z}}_1(r)$, then

$$i^2 < (2k+1)j + k^2; \qquad (1)$$

and if $p(i,j)$ lies left of $(k+1)^{th}$ run of $\mathcal{C}^{\mathbb{Z}}_1(r+1)$, then

$$i^2 < (2k+1)j + (k+1)^2. \qquad (2)$$



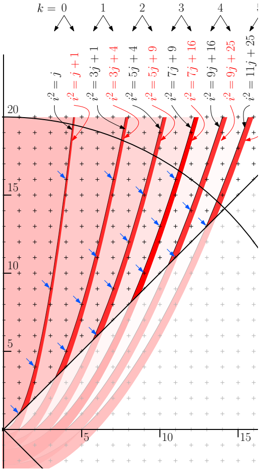
Equations 1 and 2
$\Rightarrow$ open parabolic regions (k = a constant).

$$\underline{P}_k : x^2 < (2k+1)y + k^2,$$
$$\overline{P}_k : x^2 < (2k+1)y + (k+1)^2. \qquad (3)$$

Anomalous region

$P_k := \overline{P}_k \setminus \underline{P}_k$
$= (2k+1)y + k^2 \leqslant x^2 < (2k+1)y + (k+1)^2.$

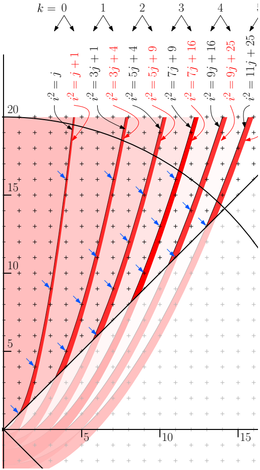
Anomalies in $\mathbb{Z}^2$



Lemma

All integer points in $F_k := P_k \cap \mathbb{Z}_1^2$ *are absentees.*

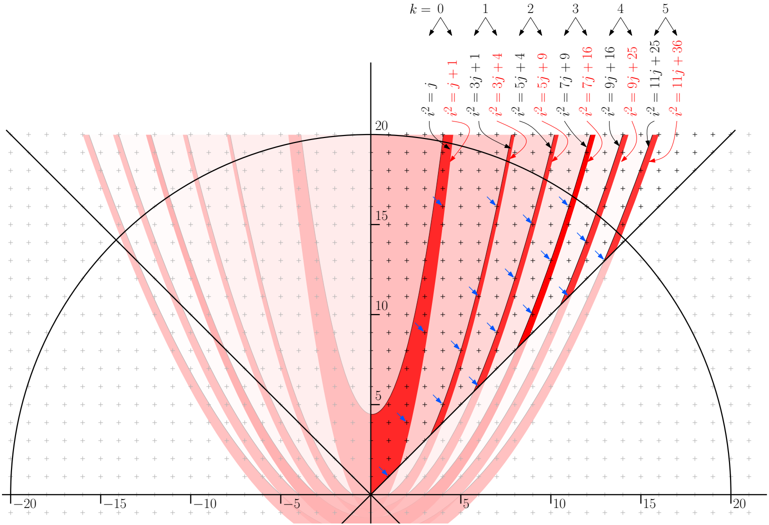
Anomalies in $\mathbb{Z}^2$



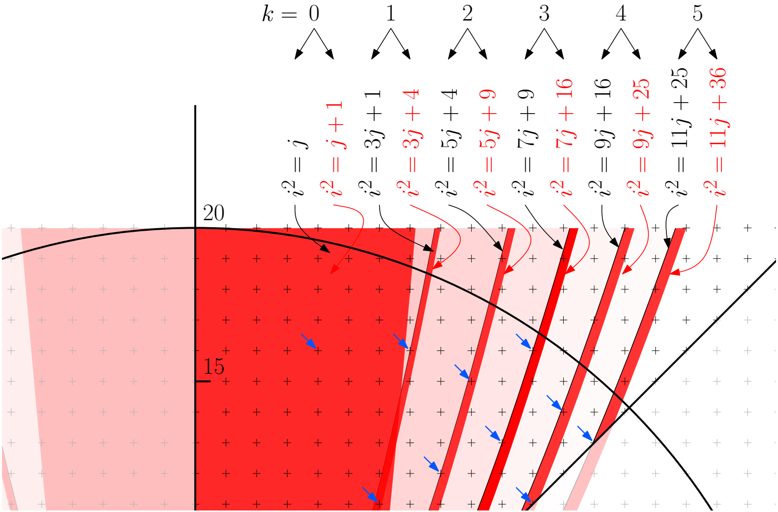
Theorem

Only and all the absentees of Octant 1 lie in $\mathcal{F} := \left\{ P_k \cap \mathbb{Z}_1^2 : k = 0, 1, 2, \ldots \right\}$.

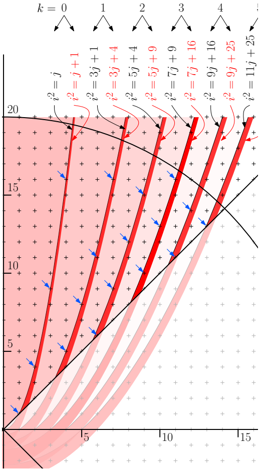
Anomalies in $\mathbb{Z}^2$



Anomalies in $\mathbb{Z}^2$



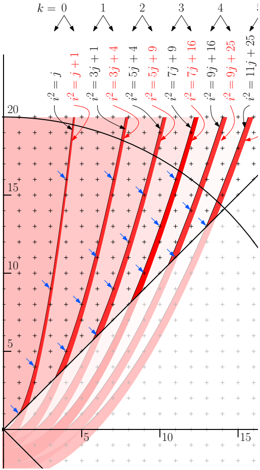
Anomalies in $\mathbb{Z}^2$



Lemma

For a given k, $P_k \cap \mathbb{Z}_1^2$ contains exactly one absentee on each vertical grid line.

Anomalies in $\mathbb{Z}^2$

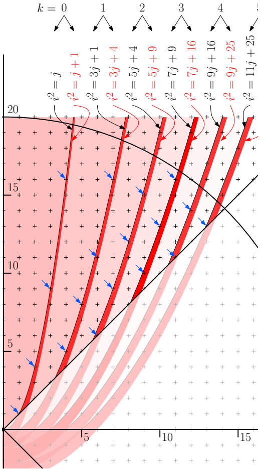


Lemma

The count of absentees contained by the parabolic strip P_k in $\mathcal{D}_1^{\mathbb{Z}}(r)$ is

$$|\mathcal{A}_k^{\mathbb{Z}}(r)| = \left\lceil \sqrt{(2k+1)r - k(k+1)} \right\rceil - \left\lceil \left((2k+1) + \sqrt{8k^2+4k+1} \right) /2 \right\rceil .$$

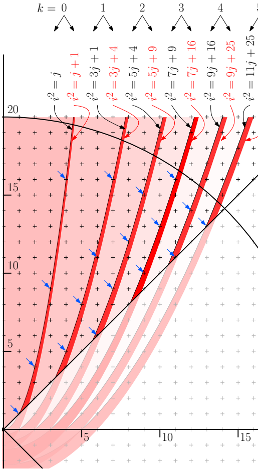
Anomalies in $\mathbb{Z}^2$



Lemma

Number of half-open parabolic strips intersecting $\mathcal{C}_1^{\mathbb{Z}}(r)$ *is*
$m_r = r - \lceil r/\sqrt{2} \rceil + 1$.

Anomalies in $\mathbb{Z}^2$

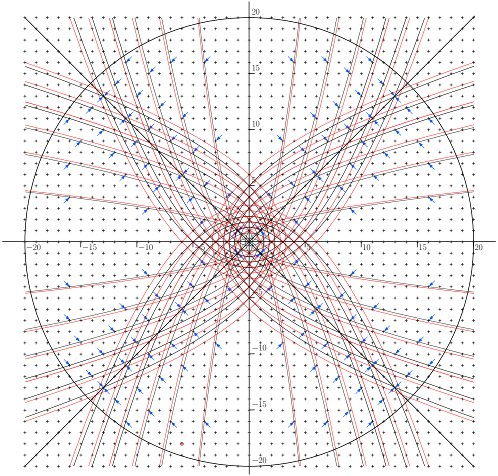


Theorem

The total count of absentees in $\mathcal{D}^{\mathbb{Z}}(r)$ is

$$|\mathcal{A}^{\mathbb{Z}}(r)| = 8 \sum_{k=0}^{m_r-1} |\mathcal{A}_k^{\mathbb{Z}}(r)| = \Theta(r^2).$$

Anomalies in $\mathbb{Z}^2$



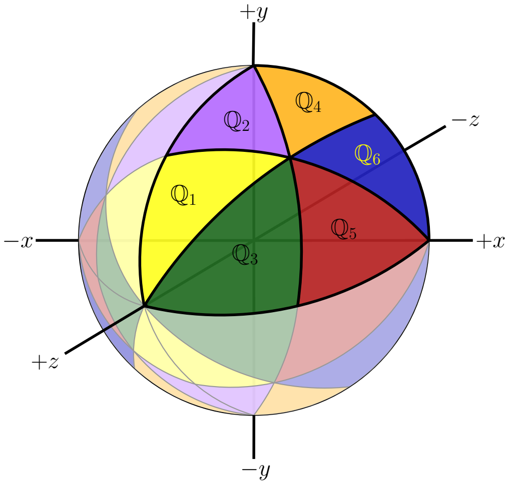
Theorem $\lim_{r\to\infty} \frac{|\mathcal{A}^{\mathbb{Z}}(r)|}{|\mathcal{D}^{\mathbb{Z}}(r)|} = 1 - \frac{2\sqrt{2}}{\pi} \approx 0.1.$

Anomalies in $\mathbb{Z}^2$

Algorithm 1.2: Disc-Absentee(r)

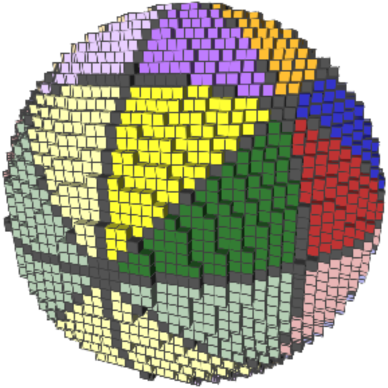
```
int i ← 0, j ← r, s ← 0, w ← r − 1, k ← 0, i_a, j_a
int l ← 2w
while j ⩾ i do
    repeat
        s ← s + 2i + 1
        i ← i + 1
    until s ⩽ w
    i_a ← i − 1, j_a ← j
    while j_a ⩾ i_a do
        if i_a^2 < (2k + 1) j_a + k^2 then
            j_a ← j_a − 1
        else
            if i_a^2 < (2k + 1) j_a + (k + 1)^2 then
                select {(i, j) : {|i|} ∪ {|j|} = {i_a, j_a}}
            i_a ← i_a − 1
    w ← w + l
    l ← l − 2, j ← j − 1, k ← k + 1
```

Sphere in $\mathbb{Z}^3$ (1)



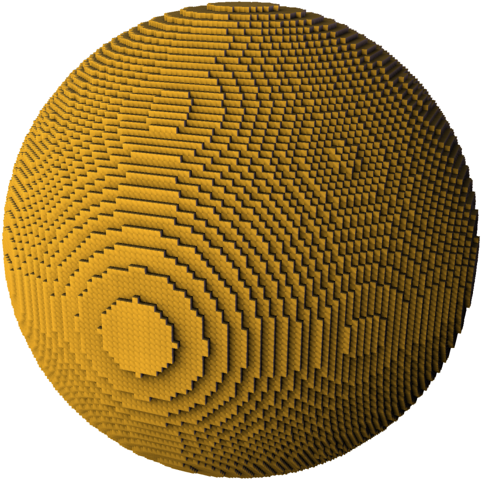
Quadraginta (48) octants of a real sphere

Sphere in $\mathbb{Z}^3$ (2)



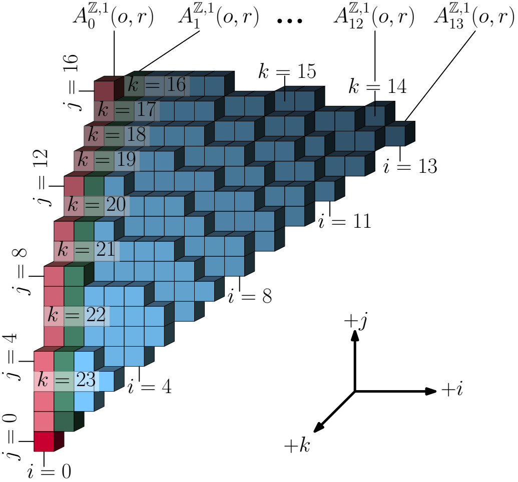
Quadraginta (48) octants of a digital sphere ($r = 23$)

Sphere in $\mathbb{Z}^3$



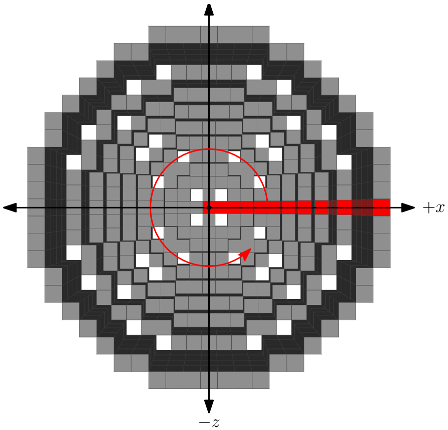
$r = 50$

Sphere in $\mathbb{Z}^3$



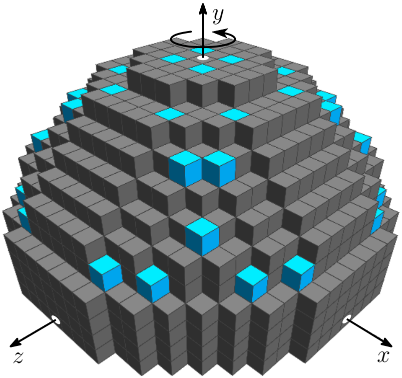
First q-octant of the digital sphere ($r = 23$)

Anomalies



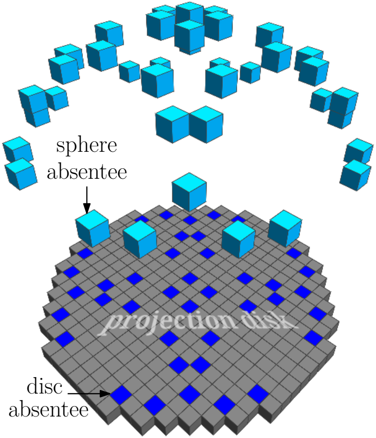
$\mathcal{H}^{\mathbb{Z}}_{\cup}(r = 10)$

Anomalies



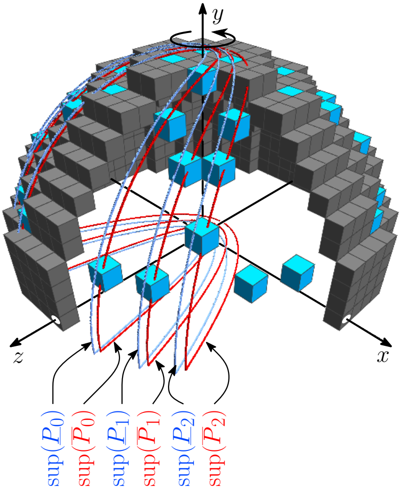
Hollow sphere absentees

Anomalies

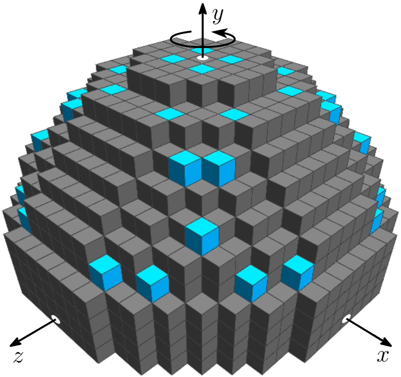


Sphere absentees $\mapsto$ disc absentees

Anomalies

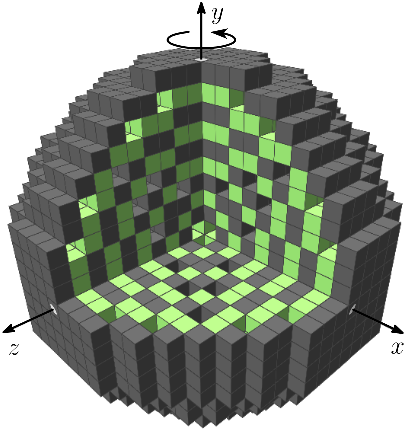


Sphere absentees in parabolic containers



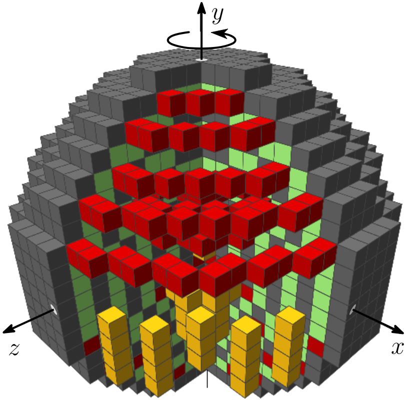
Complete (no absentee)

Anomalies



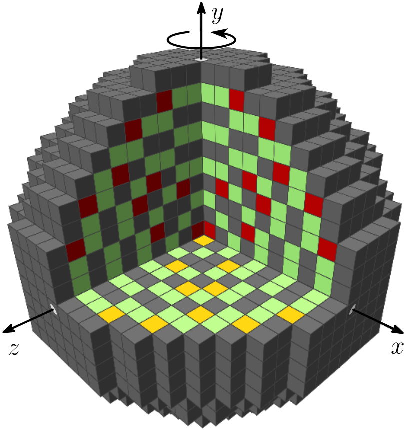
Solid sphere absentees

Anomalies



Solid sphere absentees $\mapsto$?

Anomalies



Complete (no absentee)

Thank you!