Introduction to Computational Geometry

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Outline

1. Introduction
2. Area Computation of a Simple Polygon
3. Point Inclusion in a Simple Polygon
4. Line Segment Intersection: An application of plane sweep
5. Convex Hull: An application of an incremental algorithm
6. Art Gallery Problem: A study of combinatorial geometry
Introduction

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In CG, the focus is more on discrete nature of geometric problems as opposed to continuous issues. Simply put, we would deal more with straight or flat objects (lines, line segments, polygons) or simple curved objects as circles, than with high degree algebraic curves.
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This branch of study is around thirty years old if one assumes Michael Ian Shamos’s thesis [1] as the starting point.
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CG algorithms suffer from the curse of degeneracies. So, we would make certain assumptions at times like no three points are collinear, no four points are cocircular, etc.
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Problem

Given a simple polygon $P$ of $n$ points, compute its area.
Area Computation

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Area of a convex polygon
Find a point inside $P$, draw $n$ triangles and compute the area.
Area Computation

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Find a point inside $P$, draw $n$ triangles and compute the area.

**A better idea for convex polygon**
We can **triangulate** $P$ by non-crossing diagonals into $n - 2$ triangles and then find the area.

$(n - 3)$ diagonals and $(n - 2)$ triangles
Area Computation

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Find a point inside $P$, draw $n$ triangles and compute the area.

**A better idea for convex polygon**
We can triangulate $P$ by non-crossing diagonals into $n - 2$ triangles and then find the area.

**A better idea for simple polygon**
We can do likewise.
Area Computation and Polygon Triangulation

Moral of the story

A simple polygon can be triangulated into \((n - 2)\) triangles by \((n - 3)\) non-crossing diagonals.
Area Computation and Polygon Triangulation

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A simple polygon can be \textit{triangulated} into \((n - 2)\) \textit{triangles} by \((n - 3)\) \textit{non-crossing diagonals}.

Proof

The proof is by induction on \(n\).
Area Computation and Polygon Triangulation

Moral of the story
A simple polygon can be triangulated into \((n - 2)\) triangles by \((n - 3)\) non-crossing diagonals.

Proof
The proof is by induction on \(n\).

Time complexity
We can triangulate \(P\) by a very complicated \(O(n)\) algorithm [7]
OR by a more or less simple \(O(n \log n)\) time algorithm [4].
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Point Inclusion

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Given a simple polygon $P$ of $n$ points, and a query point $q$, is $q \in P$?
Point Inclusion

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What if $P$ is convex?

Easy in $O(n)$. Takes a little effort to do it in $O(\log n)$.

$q$ is always to the left if $q \in P$, else, it varies.
Point Inclusion

Problem
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What if $P$ is convex?
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Another idea for convex polygon
- Stand at $q$ and look around the polygon.
- We can show the same result for a simple polygon also.

Total angular turn around $q$ is $2\pi$ if $q \in P$, else, 0.
Another technique: Ray Shooting

Shoot a ray and count the number of crossings with edges of $P$. If it is odd, then $q \in P$. If it is even, then $q \notin P$. Some degenerate cases need to be handled. Time taken is $O(n)$. 

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Input
A set of line segments $\mathcal{L}$ in general position in the plane. $|\mathcal{L}| = n$.

Output
Report the intersections.
Line Segment Intersection

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Line Segment Intersection

**Input**
A set of line segments $\mathcal{L}$ in general position in the plane. $|\mathcal{L}| = n$.

**Output**
Report the intersections.

**Output Sensitive Algorithm**
Number of intersections might vary from 0 to $\binom{n}{2} = O(n^2)$. So, the lower bound of the problem is $\Omega(n^2)$. The idea is now to look for an output sensitive algorithm.
An Output Sensitive Algorithm

The idea

- Avoid testing **pairs of segments** that are far apart.
An Output Sensitive Algorithm

The idea

- Avoid testing **pairs of segments** that are far apart.
- To find such pairs, imagine **sweeping a horizontal line** \( \ell \) downwards from above all segments.
An Output Sensitive Algorithm

The idea

- Avoid testing **pairs of segments** that are far apart.
- To find such pairs, imagine **sweeping** a horizontal line $\ell$ downwards from above all segments.
- Keep track of all segments that intersect $\ell$. 
An Output Sensitive Algorithm

The idea

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- $\ell$ is the sweep line and the algorithm paradigm is plane sweep.
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The idea

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- To find such pairs, imagine sweeping a horizontal line $\ell$ downwards from above all segments.
- Keep track of all segments that intersect $\ell$.
- $\ell$ is the **sweep line** and the algorithm paradigm is **plane sweep**.
- The **status** of the sweep line is the line segments intersecting it.
An Output Sensitive Algorithm

The idea

- Avoid testing pairs of segments that are far apart.
- To find such pairs, imagine sweeping a horizontal line $\ell$ downwards from above all segments.
- Keep track of all segments that intersect $\ell$.
- $\ell$ is the sweep line and the algorithm paradigm is plane sweep.
- The status of the sweep line is the line segments intersecting it.
- Only at particular points known as event points, the status needs to be updated.
Event Points and Sweep Line Status

Event Points and the Event Queue

- The start and end points of each line segment. They are static.
Event Points and Sweep Line Status

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- The start and end points of each line segment. They are static.
- The intersection points. They are dynamic and are generated as the sweep line $\ell$ sweeps down.

![Diagram showing event points and sweep line $\ell$.]
Event Points and Sweep Line Status

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- The event points are to be arranged in a data structure in a way in which the sweep line sees them.
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- The data structure should support extracting the minimum $y$-coordinate, insertion and deletion.
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- The event points are to be arranged in a data structure in a way in which the sweep line sees them.
- The data structure should support extracting the minimum $y$-coordinate, insertion and deletion.
- A heap or a balanced binary search tree can support these operations in $O(\log n)$ time.
Event Points and Sweep Line Status

**Sweep Line Status**

- We need to store the **left to right** order in which the line segments intersect $\ell$. This data structure has to be dynamic.
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- A line segment might come in (insertion) or go off (deletion) the sweep line. We need to search for its position.
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Sweep Line Status

- The sweep line status changes during three events: start and end points and intersection points and nowhere else.
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- $s_k$ and $s_l$ are two segments intersecting at a point.
Event Points and Sweep Line Status

**Sweep Line Status**

- The sweep line status changes during three events: **start** and **end** points and **intersection** points and nowhere else.

- $s_k$ and $s_l$ are two segments intersecting at a point.

- There is an event point above the intersecting point where $s_k$ and $s_l$ are adjacent and are tested for intersection. So, no intersection point is ever missed.
The Algorithm

Algorithm

- Create a heap $\mathcal{H}$ with the $y$-coordinates of end points of $\mathcal{L}$. Create sweep status data structure $\mathcal{T}$ on $x$-coordinates of the points. Initially $\mathcal{T}$ is empty.
The Algorithm

**Algorithm**

- Create a heap $H$ with the $y$-coordinates of end points of $L$. Create sweep status data structure $T$ on $x$-coordinates of the points. Initially $T$ is empty.
- Keep on extracting points from $H$ till it is non-empty.
The Algorithm

Algorithm

- Create a heap $H$ with the $y$-coordinates of end points of $L$. Create sweep status data structure $T$ on $x$-coordinates of the points. Initially $T$ is empty.

- Keep on extracting points from $H$ till it is non-empty.

- Based on the three cases: segment top end point, segment bottom end point and intersection point, take necessary actions on $T$. 
The Algorithm

Algorithm: The three cases

- **[Top end point]** Insert the line segment into $\mathcal{T}$ based on $x$-coordinates.
The Algorithm

Algorithm: The three cases

- **[Top end point]** Insert the line segment into $\mathcal{T}$ based on $x$-coordinates.
- Test for intersections with line segments to the left and right. Insert intersection point, if any, into $\mathcal{H}$.
The Algorithm

Algorithm: The three cases

- **[Top end point]** Insert the line segment into $\mathcal{T}$ based on $x$-coordinates.

- Test for intersections with line segments to the left and right. Insert intersection point, if any, into $\mathcal{H}$.

- **[Bottom end point]** Delete this line segment from $\mathcal{T}$. Test for intersections between preceding and succeeding entries in $\mathcal{T}$.
The Algorithm

**Algorithm: The three cases**

- **[Top end point]** Insert the line segment into $T$ based on $x$-coordinates.

- Test for intersections with line segments to the left and right. Insert intersection point, if any, into $H$.

- **[Bottom end point]** Delete this line segment from $T$. Test for intersections between preceding and succeeding entries in $T$.

- **[Intersection point]** Swap the line segments’ status in $T$. Check for intersections of preceding and succeeding entries.
The Analysis

Analysis

- Total number of event points is $2n + I$, where $I$ is the number of intersections.
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- The heap $H$ grows to a size at most $2n + I$. Each operation takes $O(\log(2n + I))$. As $I < n^2$, so $O(\log(2n + I)) = O(\log n)$. 
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Analysis

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- The heap $H$ grows to a size at most $2n + I$. Each operation takes $O(\log(2n + I))$. As $I < n^2$, so $O(\log(2n + I)) = O(\log n)$.

- The balanced binary search tree $T$ grows also to a size at most $2n + I$. So, each operation takes $O(\log n)$.
The Analysis

Total number of event points is $2n + I$, where $I$ is the number of intersections.

The heap $\mathcal{H}$ grows to a size at most $2n + I$. Each operation takes $O(\log(2n + I))$. As $I < n^2$, so $O(\log(2n + I)) = O(\log n)$.

The balanced binary search tree $\mathcal{T}$ grows also to a size at most $2n + I$. So, each operation takes $O(\log n)$.

So, the total time taken is $O((2n + I) \log n) = O(n \log n + I \log n)$. 
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Definition

A set \( S \subset \mathbb{R}^2 \) is convex if for any two points \( p, q \in S \), \( \overline{pq} \in S \).
Convex Hull

Definition
A set $S \subset \mathbb{R}^2$ is convex if for any two points $p, q \in S$, $\overline{pq} \in S$.

Definition
Let $\mathcal{P}$ be a set of points in $\mathbb{R}^2$. Convex hull of $\mathcal{P}$, denoted by $CH(\mathcal{P})$, is the smallest convex set containing $\mathcal{P}$.
Convex Hull Problem

Problem
Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$. 
Convex Hull Problem

Problem
Given a set of points \( P \) in the plane, compute the convex hull \( CH(P) \) of the set \( P \).

A Naive Algorithm
- Consider all line segments determined by \( \binom{n}{2} = O(n^2) \) pairs of points.
- If a line segment has all the other \( n - 2 \) points on one side of it, then it is a hull edge.
Convex Hull Problem

**Problem**

Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$.

**A Naive Algorithm**

- Consider all line segments determined by $\binom{n}{2} = O(n^2)$ pairs of points.
- If a line segment has all the other $n-2$ points on one side of it, then it is a hull edge.
- We need $\binom{n}{2}(n-2) = O(n^3)$ time.
Towards a Better Algorithm

Way forward, but how much?

- **Better characterizations lead to better algorithms.**
Towards a Better Algorithm

Way forward, but how much?

- Better characterizations lead to better algorithms.
- How much better can we make?
Towards a Better Algorithm

Way forward, but how much?

- **Better characterizations lead to better algorithms.**
- How much better can we make?
- Leads to the notion of **lower bound of a problem.**
Towards a Better Algorithm

Way forward, but how much?

- Better characterizations lead to better algorithms.
- How much better can we make?
- Leads to the notion of lower bound of a problem.
- The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$. 
Graham’s Scan: An optimal algorithm for Convex Hull

A better characterization

- Consider a walk in **clockwise** direction on the vertices of a closed polygon.
Graham’s Scan: An optimal algorithm for Convex Hull

**A better characterization**

- Consider a walk in *clockwise* direction on the vertices of a closed polygon.
- Only for a convex polygon, we will make a *right* turn always.
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The incremental paradigm
A better characterization

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The incremental paradigm

- Insert points in $P$ one by one and update the solution at each step.
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- Consider a walk in *clockwise* direction on the vertices of a closed polygon.
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The incremental paradigm
- Insert points in $P$ one by one and update the solution at each step.
- We compute the *upper hull* first. The upper hull contains the convex hull edges that bound the convex hull from above.
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The incremental paradigm

- Insert points in $P$ one by one and update the solution at each step.
- We compute the **upper hull** first. The upper hull contains the convex hull edges that bound the convex hull from above.
- Sort the points in $P$ from left to right.
Algorithm

Input: A set of points P
Algorithm

Input: A set of points P
Output: Convex Hull of P
Algorithm

Input: A set of points P
Output: Convex Hull of P
Sort P according to x-coordinate to generate
a sequence of points $p[1], p[2], \ldots, p[n]$;
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for i = 3 to n {

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Algorithm

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Output: Convex Hull of P
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for i = 3 to n {
    Append p[i] to L_U;
}
Input: A set of points P
Output: Convex Hull of P
Sort P according to x-coordinate to generate a sequence of points p[1], p[2], ..., p[n];
for i = 3 to n {
    Append p[i] to L_U;
    while(L_U contains more than two points AND the last three points in L_U do not make a right turn) {
        
    }
}
Algorithm

Input: A set of points P
Output: Convex Hull of P
Sort P according to x-coordinate to generate a sequence of points \( p[1], p[2], \ldots, p[n] \);
Put \( p[1] \) first and then \( p[2] \) in a list \( L_U \);
for \( i = 3 \) to \( n \) {
    Append \( p[i] \) to \( L_U \);
    while(\( L_U \) contains more than two points AND the last three points in \( L_U \) do not make a right turn) {
        Delete the middle of the last three points from \( L_U \);
    }
}
The Algorithm in Action

The algorithm in action

points deleted
The Algorithm

Time Complexity

- Sorting takes time $O(n \log n)$.
The Algorithm

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- The for loop is executed $O(n)$ times.
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- For each execution of the for loop, the while loop is encountered once.
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- A point once deleted, is never deleted again.
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- So, the total number of extra executions is bounded by $O(n)$.
The Algorithm

Time Complexity

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- The for loop is executed $O(n)$ times.
- For each execution of the for loop, the while loop is encountered once.
- For each extra execution of the while loop, a point gets deleted.
- A point once deleted, is never deleted again.
- So, the total number of extra executions is bounded by $O(n)$.
- Hence, the total time complexity is $O(n \log n)$. 
Proof of Correctness

The Proof of Correctness

empty region

$p_{i-1}$

$p_i$
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Art Gallery Problem

Problem
Given a simple polygon $P$ of $n$ vertices, find the minimum number of cameras that can guard $P$. 

Hardness
The above problem is NP-Hard.
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Any solution?
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**Hardness**
The above problem is NP-Hard.

**Any solution?**
- Can we find as a function of $n$ the number of cameras that suffices to guard $P$?
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Given a simple polygon $P$ of $n$ vertices, find the minimum number of cameras that can guard $P$.

Hardness
The above problem is NP-Hard.

Any solution?
- Can we find as a function of $n$ the number of cameras that suffices to guard $P$?
- Recall $P$ can be triangulated into $n - 2$ triangles. Place a guard in each triangle.
Art Gallery Problem

Can we bring the bound down?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $P$. 
Art Gallery Problem

Can we bring the bound down?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $P$.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a black, gray and white vertex.
Art Gallery Problem

Can we bring the bound down?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $P$.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a black, gray and white vertex.
- Choose the smallest color class to guard $P$. 
Art Gallery Problem

Can we bring the bound down?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $P$.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a black, gray and white vertex.
- Choose the smallest color class to guard $P$.
- Hence, $\lceil \frac{n}{3} \rceil$ guards suffice.
Art Gallery Problem

Can we bring the bound down?

- Place guards at vertices of the triangulation \( T \) of \( P \).
- We do a 3-coloring of the vertices of \( T \). Each triangle of \( T \) has a black, gray and white vertex.
- Choose the smallest color class to guard \( P \).
- Hence, \( \left\lfloor \frac{n}{3} \right\rfloor \) guards suffice.
- But, does a 3-coloring always exist?
Art Gallery Problem

A 3-coloring always exists

- Consider the dual graph $G_T$ of $T$ of $P$. 

[Diagram of a graph with vertices labeled and edges connecting them.]

Do a DFS on $G_T$ to obtain the coloring.

Place guards at those vertices that have color of the minimum color class. Hence, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are sufficient to guard $P$. 

Necessity?

Are $\left\lfloor \frac{n}{3} \right\rfloor$ guards sometimes necessary?
Art Gallery Problem

A 3-coloring always exists

- Consider the dual graph $G_T$ of $T$ of $P$.
- $G_T$ is a tree as $P$ has no holes.
A 3-coloring always exists

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Necessity?
Are $\lfloor \frac{n}{3} \rfloor$ guards sometimes necessary?
Art Gallery Theorem

The Final Theorem

For a simple polygon with $n$ vertices, $\left\lfloor \frac{n}{3} \right\rfloor$ cameras are always sufficient and occasionally necessary to have every point in the polygon visible from at least one of the cameras.
Bibliography


http://www.algorithmic-solutions.com

http://www.cgal.org