Introduction to Computational Geometry

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Outline

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- This branch of study is around thirty years old if one assumes Michael Ian Shamos's thesis [6] as the starting point.

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- Programming in CG is also a little difficult. Fortunately, libraries like LEDA [7] and CGAL [8] are now available. These libraries implement various data structures and algorithms specific to CG.
- CG algorithms suffer from the curse of degeneracies. So, we would make certain simplifying assumptions at times like no three points are collinear, no four points are cocircular, etc.

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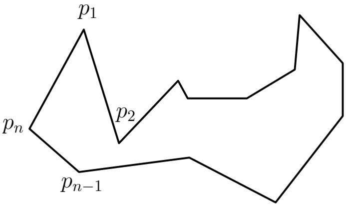
- First we consider some geometric primitives, that is, problems that arise frequently in computational geometry.

Outline

Area Computation

Problem

Given a simple polygon P of n vertices, compute its area.



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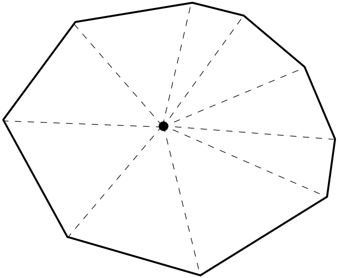
Area of a convex polygon

Find a point inside P, draw n triangles and compute the area.



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A better idea for convex polygon

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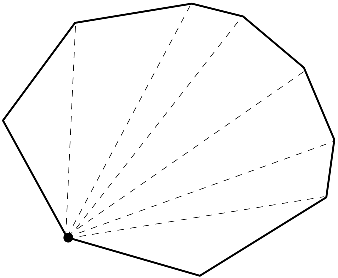

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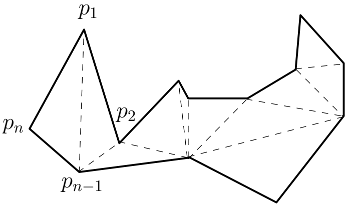
A better idea for simple polygon

We can do likewise.



Area Computation

Result

If P be a simple polygon with n vertices with coordinates of the vertex p_i being (x_i, y_i), $1 \leq i \leq n$, then twice the area of P is given by

$$2\mathcal{A}(P) = \sum_{i=1}^{n}(x_i y_{i+1} - y_i x_{i+1})$$

Polygon Triangulation

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The proof is by induction on n. □

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Time complexity

We can triangulate P by a very complicated $O(n)$ algorithm [2] OR by a *more or less simple* $O(n \log n)$ time algorithm [1].

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Point Inclusion

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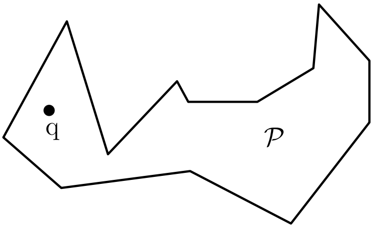
Given a simple polygon P of n points, and a query point q, is $q \in P$?



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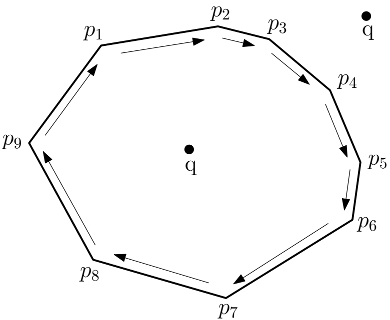
Easy in $O(n)$. Takes a little effort to do it in $O(\log n)$. Left as an exercise.



q is always to the right if $q \in \mathcal{P}$, else, it varies

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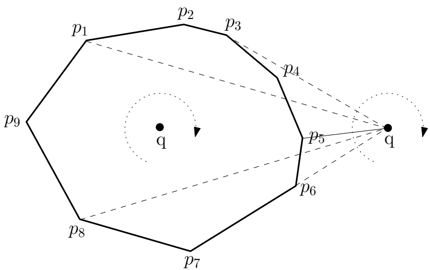
Another idea for convex polygon

Stand at q and walk around the polygon. We can show the same result for a simple polygon also.

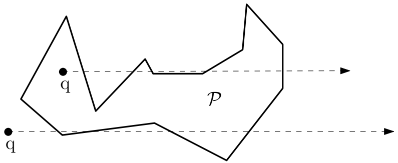


Total angular turn around q is 2π if $q \in \mathcal{P}$, else, 0

Point Inclusion

Another technique: Ray Shooting

Shoot a ray and count the number of crossings with edges of P. If it is odd, then $q \in P$. If it is even, then $q \notin P$. Some degenerate cases need to be handled. Time taken is $O(n)$.



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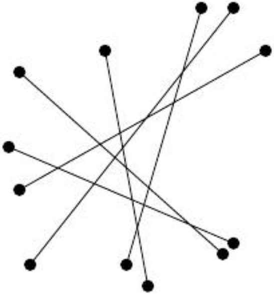
Line Segment Intersection

Input

A set of line segments $\mathcal{L}$ in general position in the plane. $|\mathcal{L}| = n$.



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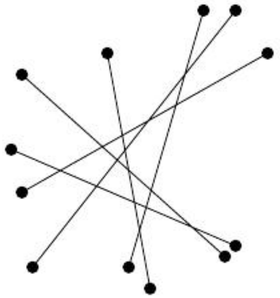

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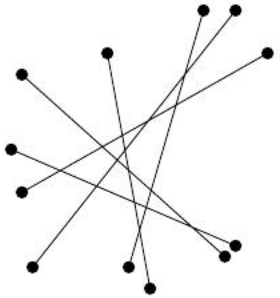
Output Sensitive Algorithm

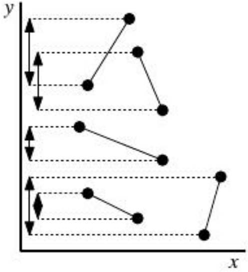
Number of intersections might vary from 0 to $\binom{n}{2} = O(n^2)$. So, the lower bound of the problem is $\Omega(n^2)$. The idea is now to look for an output sensitive algorithm.



An Output Sensitive Algorithm

The idea

- Avoid testing pairs of segments that are far apart.



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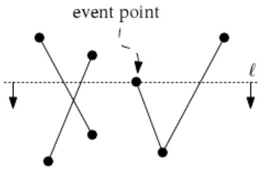
- Avoid testing pairs of segments that are far apart.
- To find such pairs, imagine sweeping a horizontal line ℓ downwards from above all segments.



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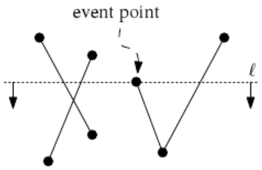

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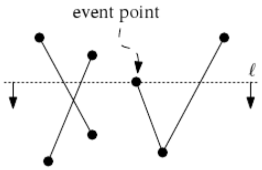
- ℓ is the sweep line and the algorithm paradigm is plane sweep.



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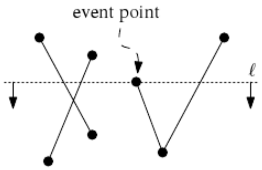

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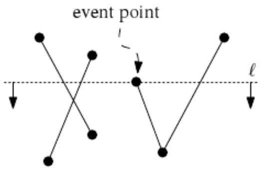
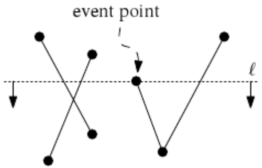
- Only at particular points known as event points, the status needs to be updated.



Event Points and Sweep Line Status

Event Points and the Event Queue

- The start and end points of each line segment. They are static.



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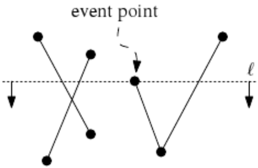

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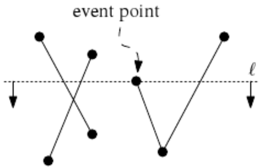

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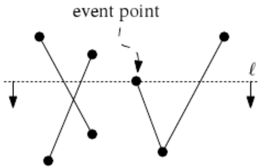

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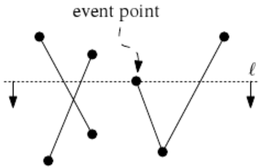

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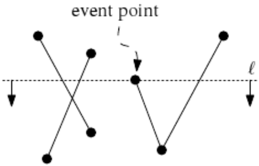

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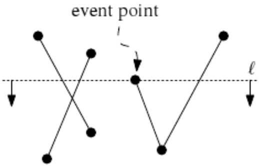

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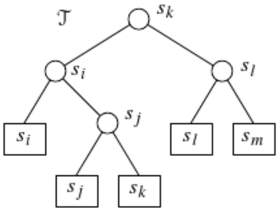

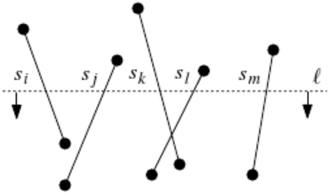

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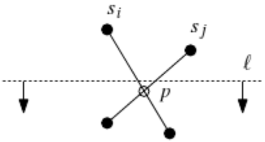

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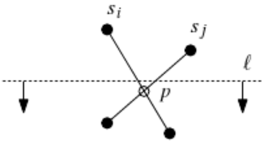

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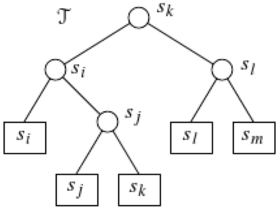
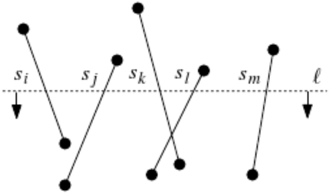
- s_i and s_j are two segments intersecting at p.
- There is an event point above p where s_i and s_j are adjacent and are tested for intersection. So, no intersection point is ever missed.

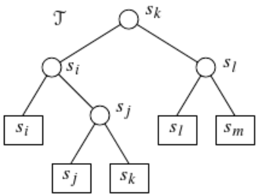


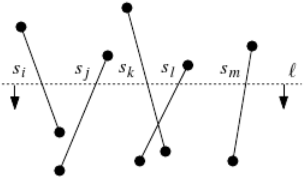


The Algorithm

Algorithm

- Create a heap $\mathcal{H}$ with the y-coordinates of end points of $\mathcal{L}$. Create sweep status data structure $\mathcal{T}$ on x-coordinates of the points. Initially $\mathcal{T}$ is empty.





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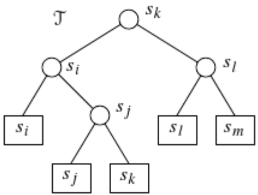

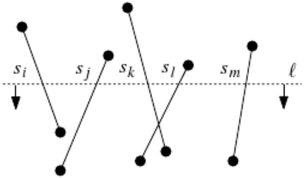

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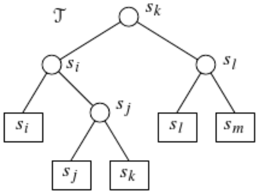
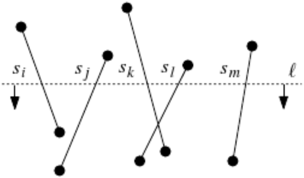
- Based on the three cases: segment top end point, segment bottom end point and intersection point, we take necessary actions.





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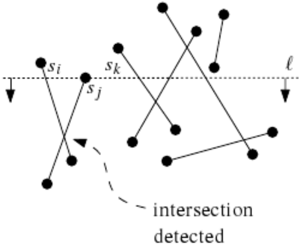

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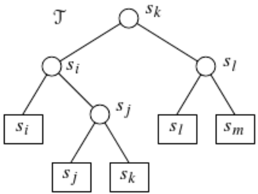

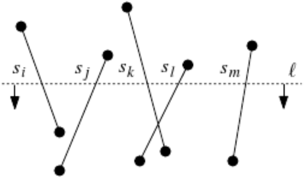

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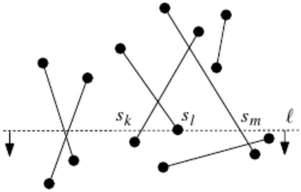

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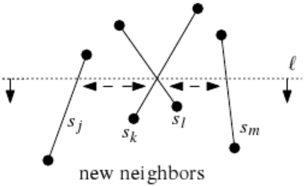
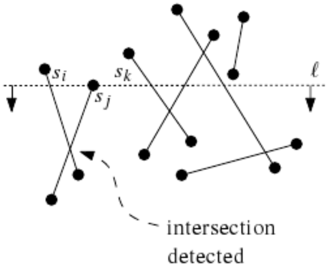
- **[Intersection point]** Swap the line segments' status in $\mathcal{T}$. Check for intersections of preceding and succeeding entries.



The Analysis

Analysis

- The heap $\mathcal{H}$ grows to a size at most $2n + I$ where I is the number of intersections. Each operation takes $O(\log(2n + I))$. As $I \leq n^2$, so $O(\log(2n + I)) = O(\log n)$.



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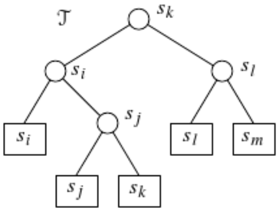

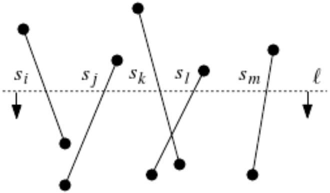

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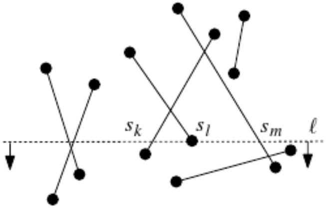
- So, the total time taken is $O((2n + I)\log n) = O(n \log n + I \log n)$.



Outline

Definitions

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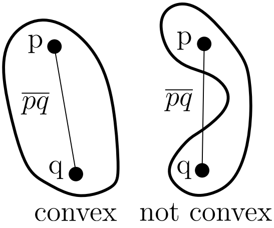

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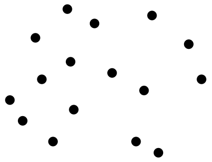

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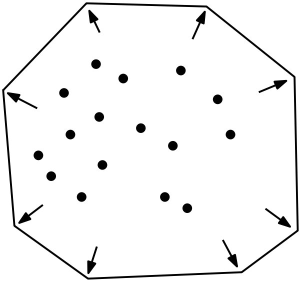

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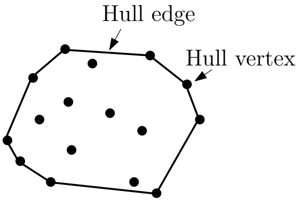

Convex Hull Problem

Problem

Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$.

A Naive Algorithm

Outline

- Consider all line segments determined by $\binom{n}{2} = O(n^2)$ pairs of points.

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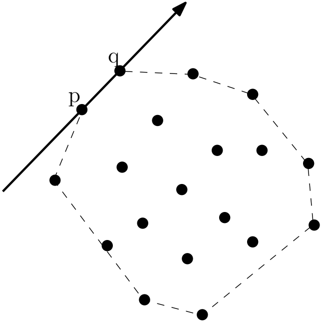

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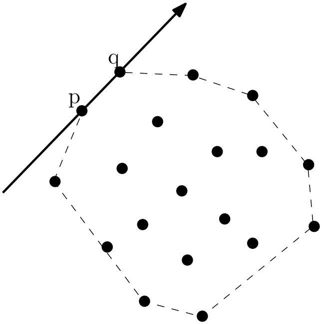
- We need $\binom{n}{2}(n-2) = O(n^3)$ time.



Towards a Better Algorithm

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- Better characterizations lead to better algorithms.

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- Better characterizations lead to better algorithms.
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- Leads to the notion of lower bound of a problem.
- The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$.

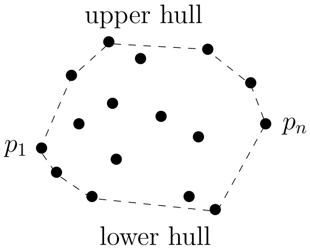
Optimal Algorithms

- Grahams scan, time complexity $O(nlogn)$.
(Graham, R.L., 1972)
- Divide and conquer algorithm, time complexity $O(nlogn)$.
(Preparata, F. P. and Hong, S. J., 1977)
- Jarvis's march or gift wrapping algorithm, time complexity $O(nh)$ where h number of vertices of the convex hull.
(Jarvis, R. A., 1973)
- Most efficient algorithm to date is based on the idea of Jarvis's march, time complexity $O(nlogh)$.
(T. M. Chan, 1996)

Definitions

A better characterization

- Consider a walk in clockwise direction on the vertices of a closed polygon.



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- Consider a walk in clockwise direction on the vertices of a closed polygon.
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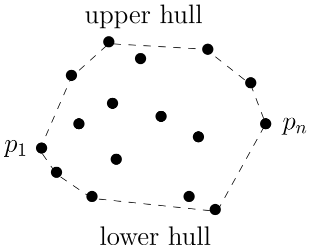

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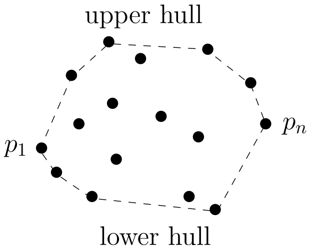

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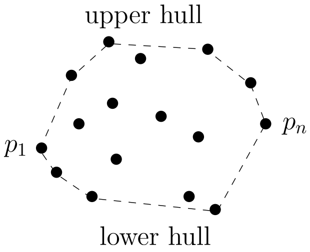

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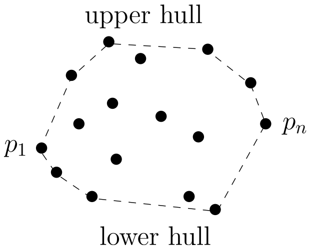

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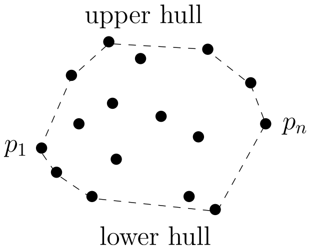
- Sort the points in $\mathcal{P}$ from left to right.



A Naive Algorithm

`Input: A set P of n points in the plane`

A Naive Algorithm

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Input: A set P of n points in the plane
Output: Convex Hull of P
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A Naive Algorithm

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Input: A set P of n points in the plane
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Sort P according to x-coordinate to generate
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         Delete the middle of the last
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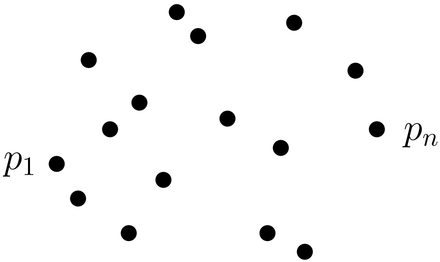
The Algorithm in Action



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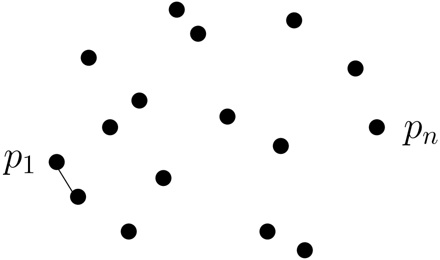

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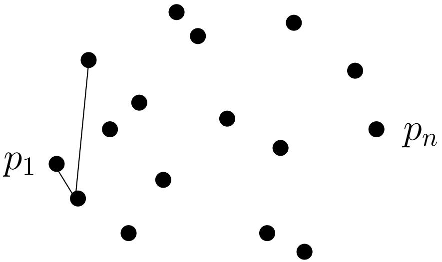

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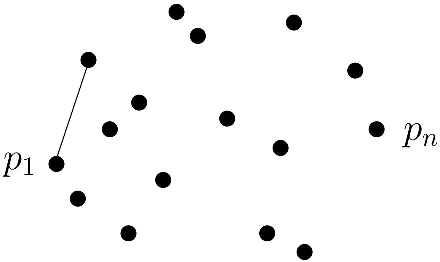

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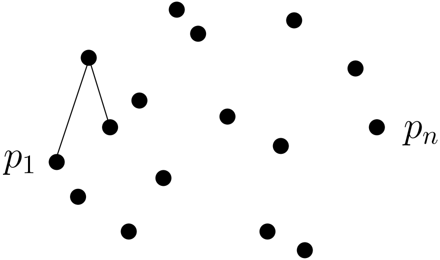

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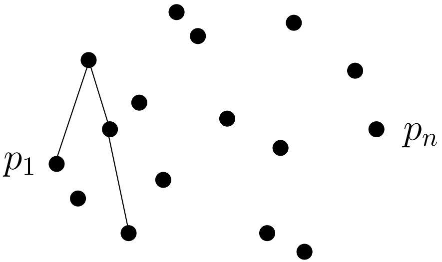

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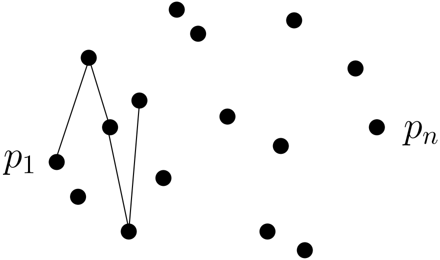

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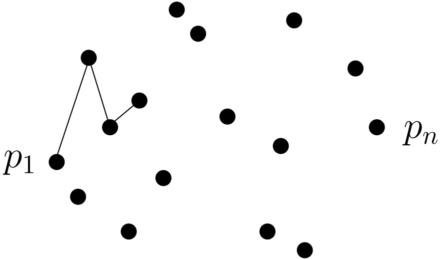

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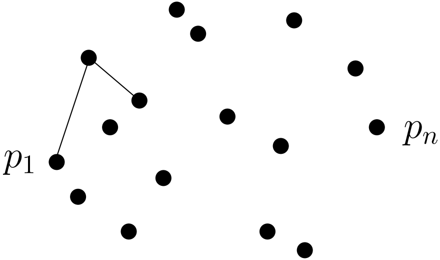

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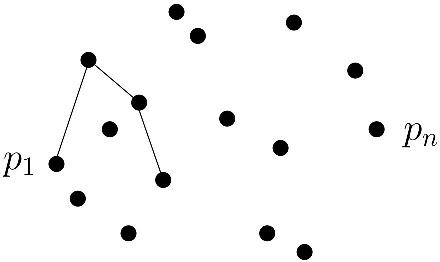

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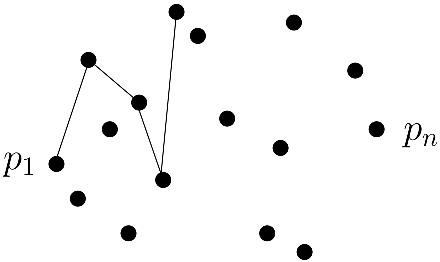

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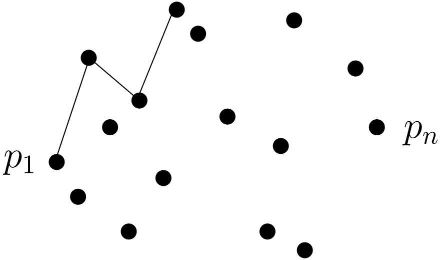

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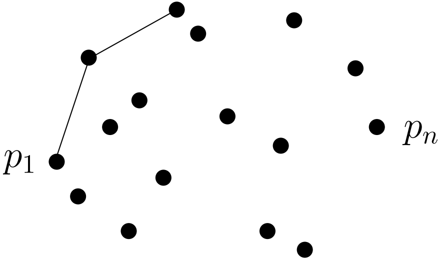

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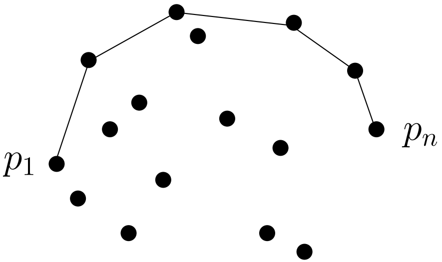

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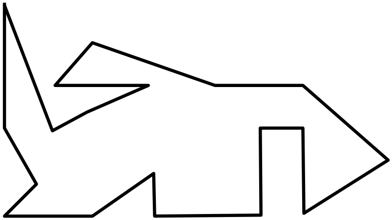
Outline

Art Gallery Problem

The problem

Given a simple polygon $\mathcal{P}$ of n vertices, find the minimum number of cameras that can guard $\mathcal{P}$.



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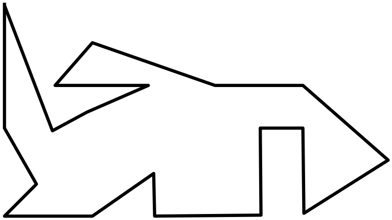

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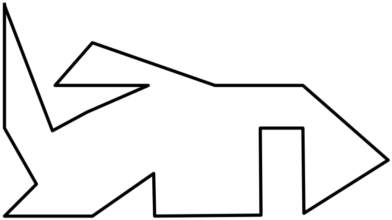

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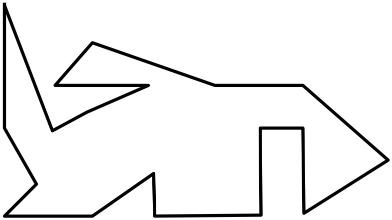

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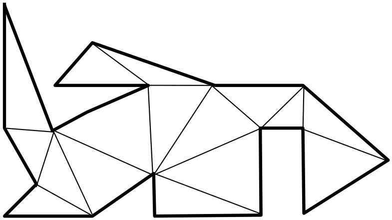
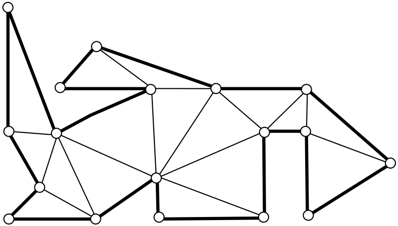
- Recall $\mathcal{P}$ can be triangulated into $n-2$ triangles. Place a guard in each triangle.



Art Gallery Problem

Can the bound be reduced?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $\mathcal{P}$.



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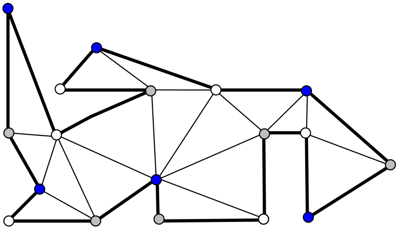

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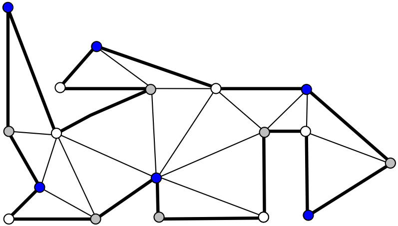

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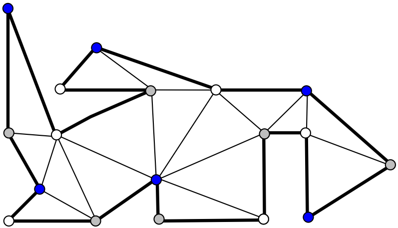

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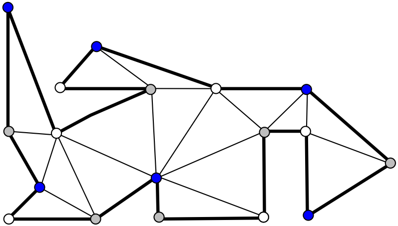
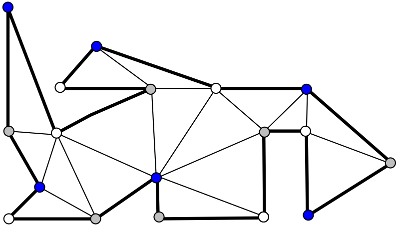
- But, does a 3-coloring always exist?



Art Gallery Problem

A 3-coloring always exist



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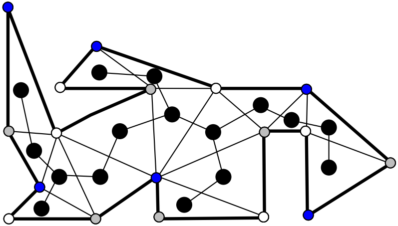
- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.



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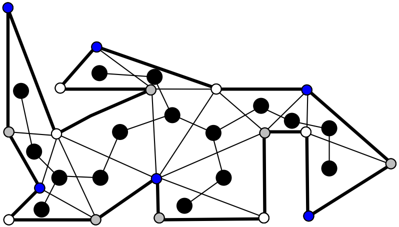
- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.



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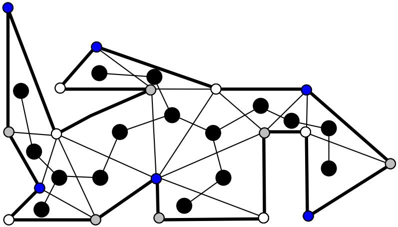
- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.
- Do a DFS on $\mathcal{G}_{\mathcal{T}}$ to obtain the coloring.



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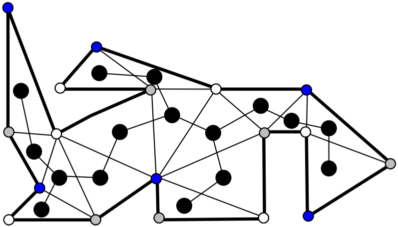
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.
- Do a DFS on $\mathcal{G}_{\mathcal{T}}$ to obtain the coloring.
- Place guards at those vertices that have color of the minimum color class. Hence, $\lfloor \frac{n}{3} \rfloor$ guards are sufficient to guard $\mathcal{P}$.



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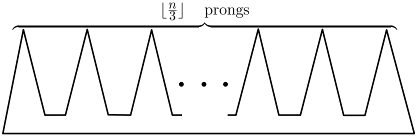
Necessity?

Are $\lfloor \frac{n}{3} \rfloor$ guards sometimes necessary?



Art Gallery Theorem

Final Result

For a simple polygon with n vertices, $\lfloor \frac{n}{3} \rfloor$ cameras are always sufficient and occasionally necessary to have every point in the polygon visible from at least one of the cameras.

References I

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References II

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Thank you!