Geometric Graphs

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Workshop on Introduction to Graph and Geometric Algorithms
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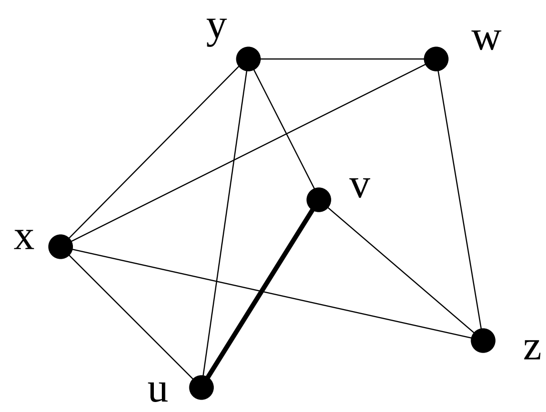
Geometric Graph



- ☆ V = set of geometric objects (point set in the plane)
- ☆ $E = \{(u, v)\}$ based on some geometric condition

Questions on Geometric Graphs

- Problems on graphs
  - Independent set, coloring, clique, etc.
- Combinatorial/Structural questions
  - Obtain Bounds
  - Characterization
- Computational questions
  - Efficient Algorithm
  - Approximation

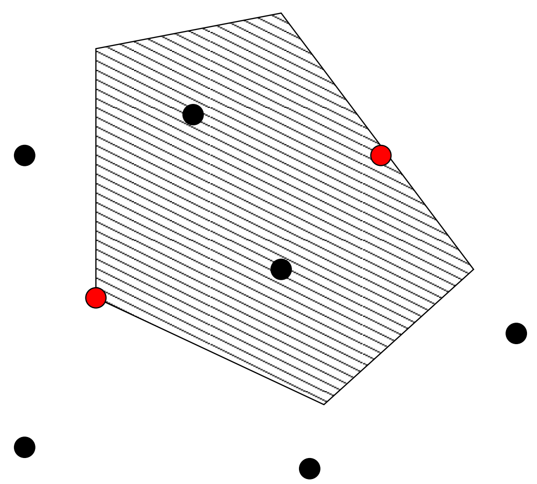
Geometric graphs

- V - set of geometric objects
- E - object i and j satisfy certain geometric condition

- Broad classes of geometric graphs (based on edge condition)
  - Proximity graphs
  - Intersection graphs
  - Distance based graphs

Proximity Graphs

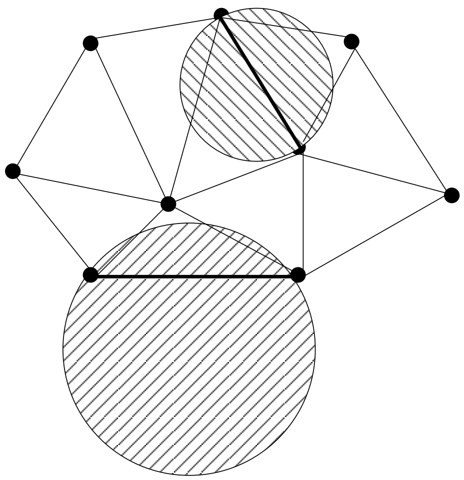
- ☆ P - point set in plane
- ☆ $R_{i,j}$ - proximity region defined by i and j



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $R_{i,j}$ is empty
- ☆ Examples - Delaunay, Gabriel, Relative Neighborhood Graph
- ☆ Applications - Graphics, wireless networks, GIS, computer vision, etc.

Delaunay Graph - Classic Example

- ☆ P - point set in plane



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $\exists$ some empty circle thro' i and j
- ☆ Triangle (i, j, k) if circumcircle(i, j, k) is empty
  (Equivalent condition)
- ☆ Applications - Graphics, mesh generation, computer vision, etc.

Questions on Delaunay Graph

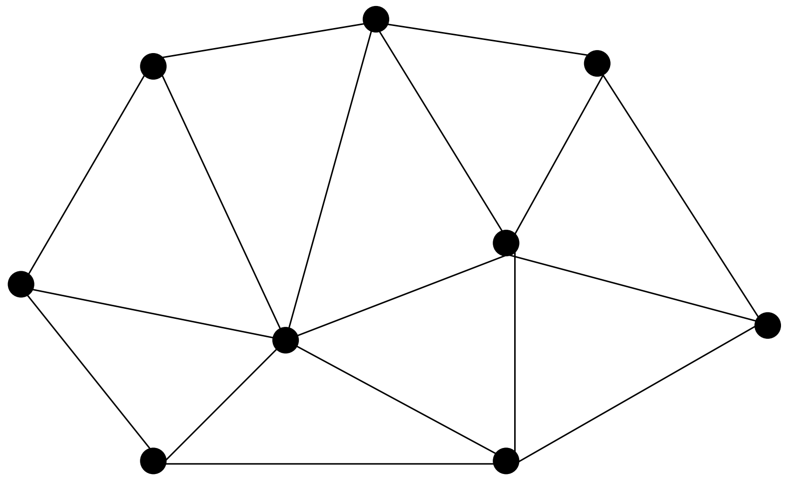
- Combinatorial - Bounds on
  - Maximum size of edge set?
  - Chromatic number?
  - Maximum independent set?

  (Over all possible point sets P)

- Computational
  - Efficient Algorithm

Delaunay Graph - Classic Example

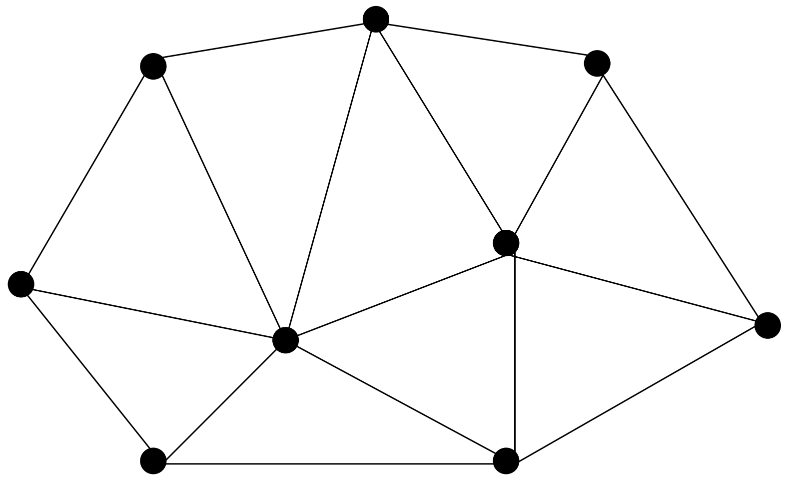
⋆ P - point set in plane



⋆ Observations:

Delaunay Graph - Classic Example

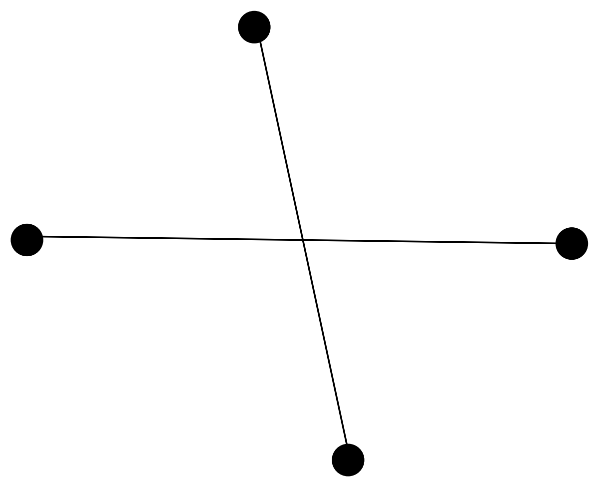
☆ P - point set in plane



☆ Observations: Planar?

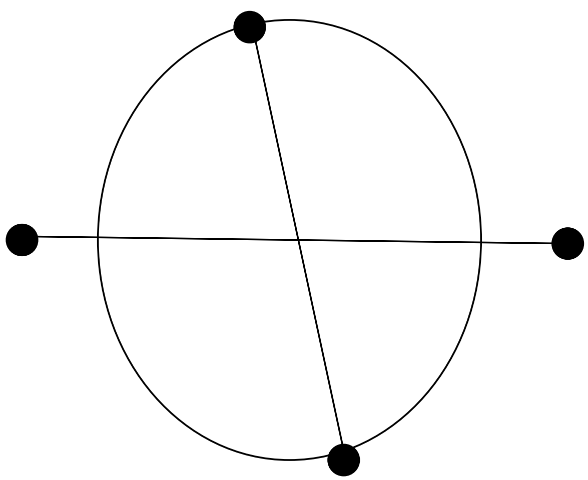
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



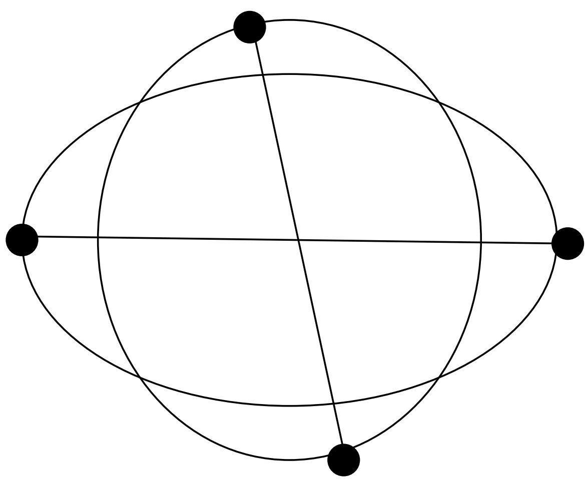
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



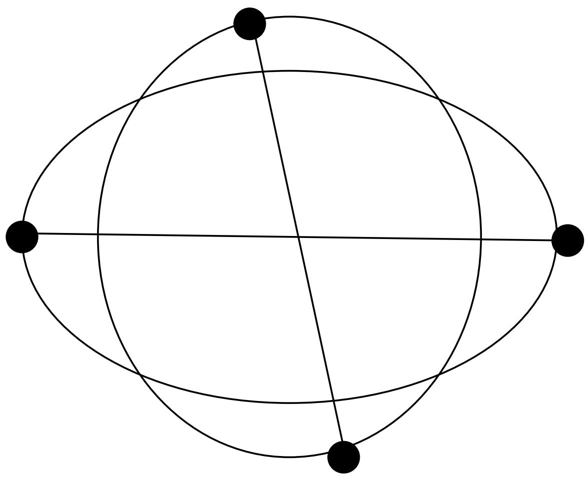
Delaunay Graph - Planar

⋆ Let, if possible, 2 edges cross



Delaunay Graph - Planar

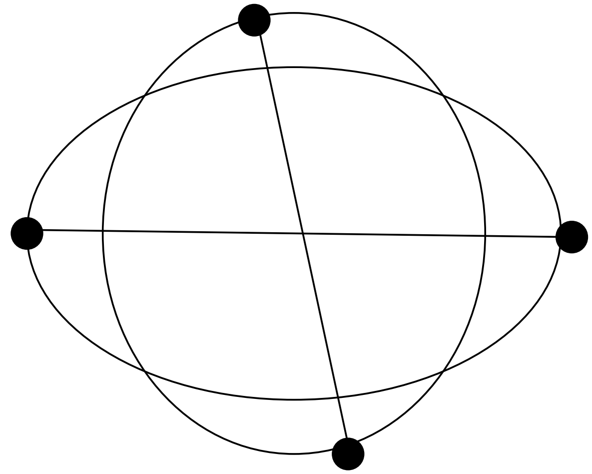
⋆ Let, if possible, 2 edges cross



⋆ Circles c'ant intersect like this (why?)

Delaunay Graph - Planar

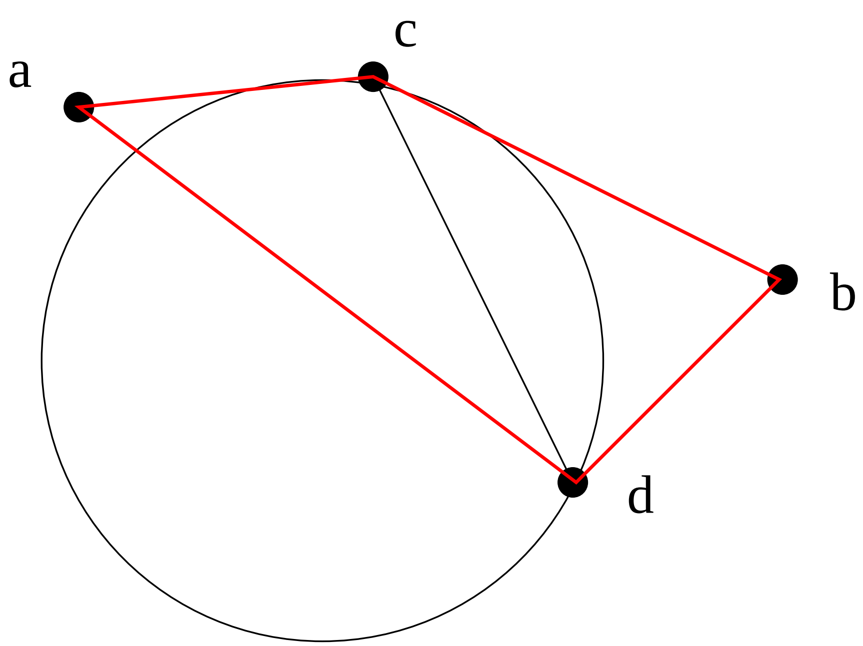
- Let, if possible, 2 edges cross



- Circles c'ant intersect like this (why?)
- One endpoint of an edge lies within the other circle
  - Contradiction

Delaunay Graph - Proof using angles

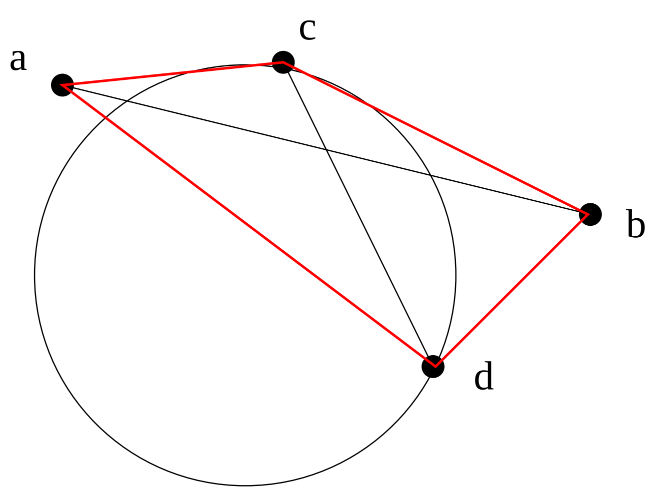
- Consider any circle passing through c and d
- Points a and b are outside the circle



- $\angle cad + \angle cbd < 180$

Delaunay Graph - Proof using angles

- ☆ Let, if possible, edges ab and cd cross
- ☆ Consider the quadrilateral $acdb$



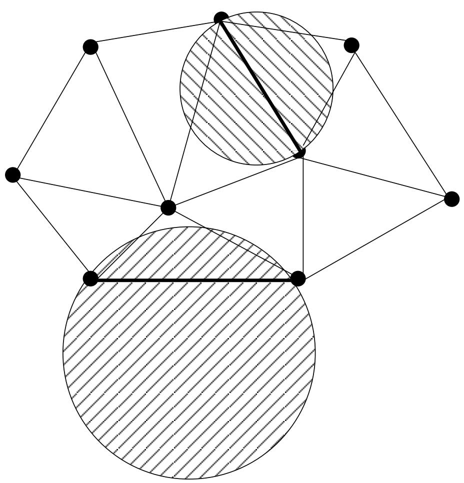
- ☆ cd is an edge $\implies \angle cad + \angle cbd < 180$
- ☆ ab is an edge $\implies \angle acb + \angle adb < 180$
- ☆ $\angle cad + \angle cbd + \angle acb + \angle adb < 360$
  - Contradiction

Questions on Delaunay Graph

- Given any n-point set P in the plane
  - Delaunay graph is planar
- Maximum size of edge set
  - $\leq 3n - 6$ (Euler's formula)
- Chromatic number
  - ≤ 4 (Four color theorem)
- Maximum independent set
  - $\geq n/4$ (Chromatic number)

Delaunay Graph

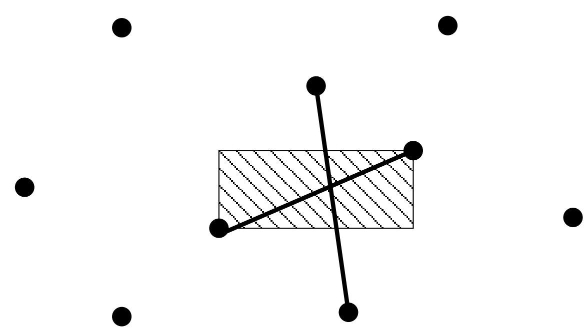
- ☆ P - point set in plane



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $\exists$ some empty **circle** thro' i and j

Delaunay Graph - Variants

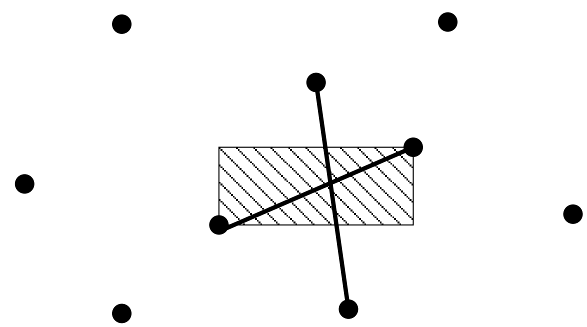
- ⋆ Edges defined by other objects (instead of circles)
- ⋆ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j



- ⋆ Bounds on the size of maximum independent set?
- ⋆ Application: Frequency assignment in wireless networks

Delaunay Graph wrt Rectangles

- ⋆ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j



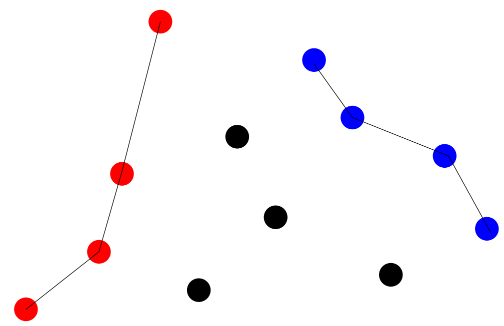
- ⋆ Graph Properties
  - Graph can have $\Omega(n^2)$ edges
  - $K_n, n \geq 5$ is a forbidden subgraph

Bounds on Independent Set Size

Theorem: Any Delaunay graph (wrt rectangles) has an independent set of size atleast $\sqrt{n}/2$

Bounds on Independent Set Size

- Same slope sequence of points
  - +ve slope sequence (Red)
  - -ve slope sequence (Blue)



- Same slope sequence of size $2k$
  - Independent set of size k

Bounds on Independent Set Size

Erdos-Szekeres Theorem: Let P be any set of $m^2 + 1$ points in the plane. There exists a same slope sequence (+ve or -ve) of size $m + 1$.

- ☆ Atleast six different proofs
  (Monotone subsequence survey by Michael Steele)

- ☆ Let S be any sequence of $m^2 + 1$ integers. There exists a monotonic subsequence (increasing or decreasing) of size $m + 1$.

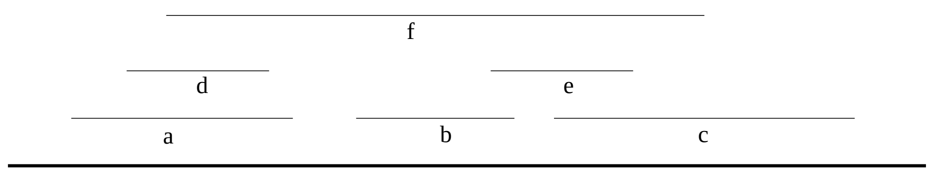
Independent Set - Open Problem

- Size of maximum independent set - Lower bound
  - $\Omega(n^{0.5})$ (Slope sequence)
  - Improved to $\Omega(n^{0.618-\epsilon})$ (Ajwani et al, SPAA '07)
- Size of maximum independent set - Upper bound
  - $O(n/\log n)$ (Pach et al '08)
- Conjecture: Close to $O(n/\log n)$
- Open problem : Obtain better upper/lower bounds

Intersection Graphs

- ☆ Interval Graph - Classic example

- ☆ S - set of geometric objects s_i (intervals on the real line)

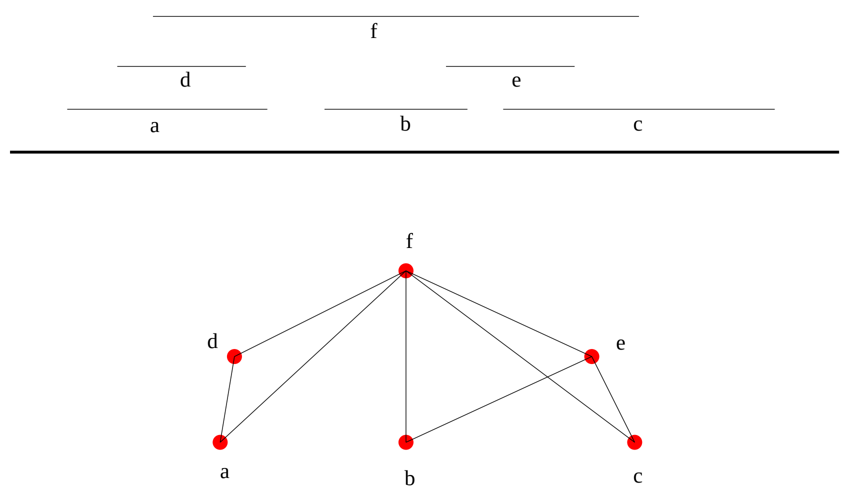


- ☆ V - set of object s_i

- ☆ $(s_i, s_j) \in E$ if objects s_i and s_j intersect

Interval Graphs

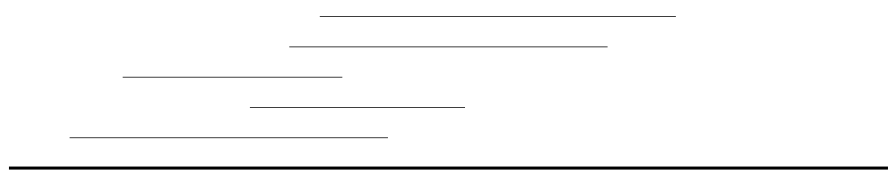
- S - set of intervals on the line



- V - set of object s_i
- $(s_i, s_j) \in E$ if objects s_i and s_j intersect
- Graph problems - Maximum independent set, Maximum clique, Chromatic number, etc.
  - Can be computed efficiently

Intervals

- ☆ S - set of intervals on the real line
- ☆ Every 2 intervals in S intersect



- ☆ Claim: All the intervals have a common intersection

Intervals

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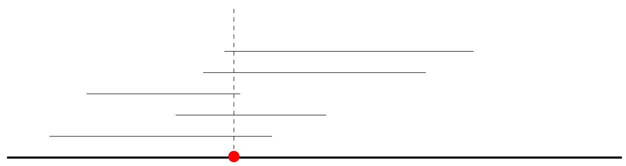
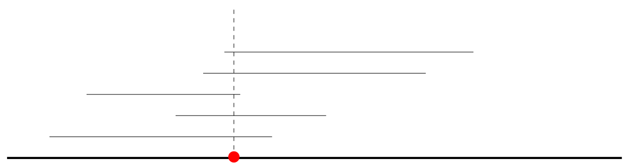


- ☆ Claim: All the intervals have a common intersection

Intervals

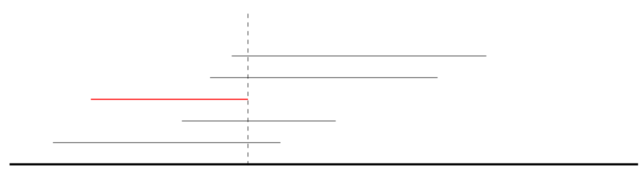
- ⋆ S - set of intervals on the real line
- ⋆ Every 2 intervals in S intersect
- ⋆ Claim: All the intervals have a common intersection



- ⋆ Induction proof (Exercise)
- ⋆ Constructive proof
  - Construct a point p that is contained in all the intervals

Intervals

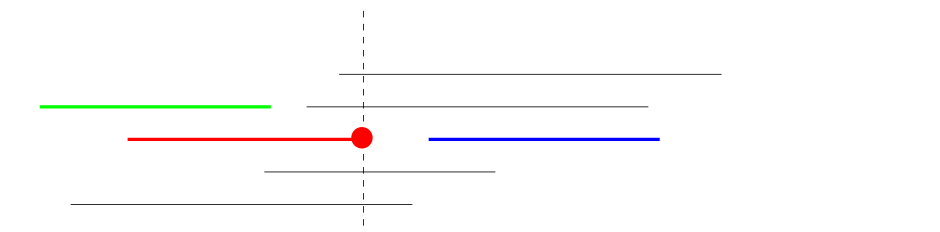
- S - set of intervals on the real line
- Every 2 intervals intersect
- Constructive proof
  - Construct a point p that is contained in all the intervals
- p : Right endpoint of interval that ends leftmost
  - Leftmost right endpoint



- Claim: All the intervals contain p

Intervals

- ☆ Construct a point p that is contained in all the intervals
- ☆ p : Right endpoint of interval that ends leftmost
  - Leftmost right endpoint
- ☆ Claim: All the intervals contain p
- ☆ Proof by contradiction



Intersection Graphs of Axis Parallel Rectangles

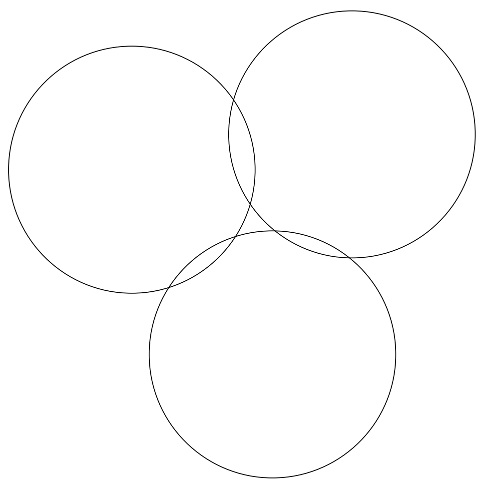
- S - set of axis parallel rectangles
- Every 2 rectangles intersect
  - Claim: There exists a point p contained in all the rectangles
  - Is it true?

Intersection Graphs of Circles

- S - set of circles
- Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles

Intersection Graphs of Circles

- S - set of circles
- Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles
  - Not true

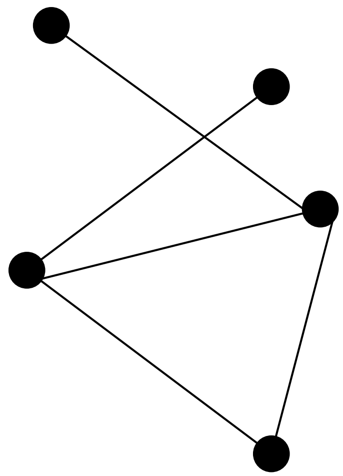


Intersection Graphs of Circles

- ☆ S - set of circles
- ☆ Every 2 circles intersect
  - Claim: There exists a point p contained in all the circles
  - Not true
- ☆ Every 3 circles intersect
  - Claim: There exists a point p contained in all the circles
  - True
- ☆ Helly Theorem: Statement true for convex objects

Distance based Graphs

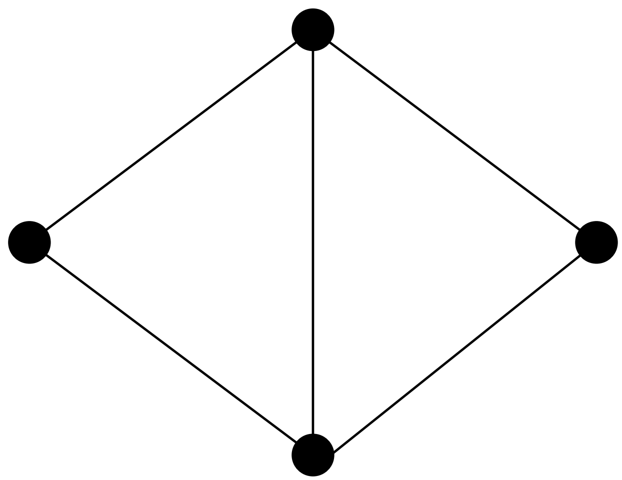
- Unit distance graph
  - V - point set in plane
  - $(i, j) \in E$ if $d(i, j) = 1$



- Place points so as to maximize the number of edges
- Can you get a complete graph? (even for $n = 4$)

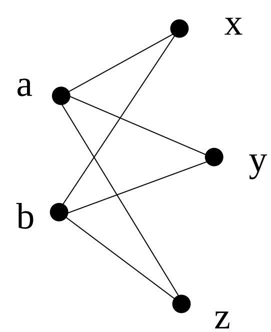
Distance based Graphs

- Unit distance graph
  - V - point set in plane
  - $(i, j) \in E$ if $d(i, j) = 1$



Unit Distance Graph

- V - point set P
- $(i,j) \in E$ if $d(i,j) = 1$
- Maximum number of edges? (Erdos)
  - Over all possible n-point set
- $O(n^{3/2})$ edges
  - Forbidden $K_{2,3}$



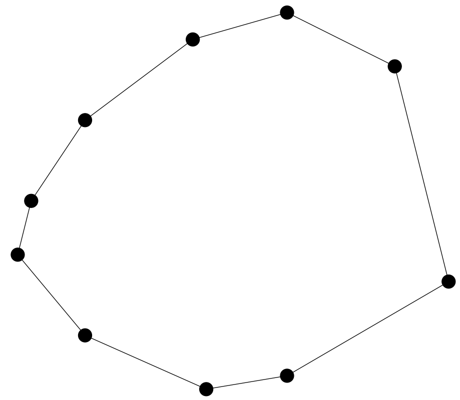
- $O(n^{4/3})$ edges
  - Crossing Lemma, Cuttings, Arrangement of Circles

Unit Distance Graph - Open Problem

- Upper bound
  - $O(n^{4/3})$ edges
- Lower bound
  - $\Omega(n^{1+\frac{c}{\log \log n}})$ [Erdos]
- Conjecture: Lower bound is tight

Unit Distance Graph - Convex Point Set

- ⋆ Convex Point Set



- ⋆ Upper bound: $O(n \log n)$ edges

- ⋆ Lower bound: $2n - 7$ edges

- ⋆ Conjecture: Lower bound is tight ($2n$ edges)

Questions

Questions