Introduction to Computational Geometry

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Outline

1. Introduction

2. Area Computation of a Simple Polygon

3. Point Inclusion in a Simple Polygon

4. Convex Hull: An application of incremental algorithm

5. Art Gallery Problem: A study of combinatorial geometry
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This branch of study is around thirty years old if one assumes Michael Ian Shamos’s thesis [6] as the starting point.
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- Programming in CG is a little difficult. Fortunately, libraries like LEDA [7] and CGAL [8] are now available. These libraries implement various data structures and algorithms specific to CG.
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CG algorithms suffer from the curse of degeneracies. So, we would make certain simplifying assumptions at times like no three points are collinear, no four points are cocircular, etc.
In this lecture, we touch upon a few simple topics for having a glimpse of the area of computational geometry.
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First we consider some geometric primitives, that is, problems that arise frequently in computational geometry.
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Area Computation

Problem
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Area Computation

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**Area of a convex polygon**

Find a point inside \( P \), draw \( n \) triangles and compute the area.
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A better idea for convex polygon
We can triangulate $P$ by non-crossing diagonals into $n - 2$ triangles and then find the area.
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A better idea for simple polygon
We can do likewise.
Area Computation

Result

If $P$ be a simple polygon with $n$ vertices with coordinates of the vertex $p_i$ being $(x_i, y_i)$, $1 \leq i \leq n$, then twice the area of $P$ is given by

$$2A(P) = \sum_{i=1}^{n} (x_i y_{i+1} - y_i x_{i+1})$$
Theorem

Any simple polygon can be triangulated.
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**Theorem**

*A simple polygon can be *triangulated* into \((n - 2)\) *triangles* by \((n - 3)\) *non-crossing diagonals*. 

**Proof.**
The proof is by induction on \(n\).

**Time complexity**

We can triangulate \(P\) by a very complicated \(O(n)\) algorithm [2] or by a more or less simple \(O(n \log n)\) time algorithm [1].
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Polygon Triangulation

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Point Inclusion

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**What if $P$ is convex?**
Easy in $O(n)$. Takes a little effort to do it in $O(\log n)$. Left as an exercise.

$q$ is always to the right if $q \in P$, else, it varies
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Another idea for convex polygon
Stand at $q$ and walk around the polygon. We can show the same result for a simple polygon also.

Total angular turn around $q$ is $2\pi$ if $q \in P$, else, 0
Another technique: Ray Shooting

Shoot a ray and count the number of crossings with edges of \( P \). If it is odd, then \( q \in P \). If it is even, then \( q \notin P \). Some degenerate cases need to be handled. Time taken is \( O(n) \).
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![Convex and non-convex examples](image-url)
Definitions

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A set $S \subset \mathbb{R}^2$ is convex if for any two points $p, q \in S$, the line segment $pq \in S$.

Definition
Let $\mathcal{P}$ be a set of points in $\mathbb{R}^2$. Convex hull of $\mathcal{P}$, denoted by $CH(\mathcal{P})$, is the smallest convex set containing $\mathcal{P}$. 
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Convex Hull Problem

Problem

Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$. 
A Naive Algorithm

Outline

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- If a line segment has all the other \( n - 2 \) points on one side of it, then it is a hull edge.
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- Consider all line segments determined by $\binom{n}{2} = O(n^2)$ pairs of points.
- If a line segment has all the other $n - 2$ points on one side of it, then it is a hull edge.
- We need $\binom{n}{2}(n - 2) = O(n^3)$ time.
Towards a Better Algorithm

How much betterment is possible?

- Better characterizations lead to better algorithms.
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The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$. 
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- The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$. 
Optimal Algorithms

- **Grahams scan**, time complexity $O(n \log n)$.
  (Graham, R.L., 1972)

- **Divide and conquer algorithm**, time complexity $O(n \log n)$.
  (Preparata, F. P. and Hong, S. J., 1977)

- **Jarvis’s march or gift wrapping algorithm**, time complexity $O(nh)$ where $h$ is the number of vertices of the convex hull.
  (Jarvis, R. A., 1973)

- Most efficient algorithm to date is based on the idea of Jarvis’s march, time complexity $O(n \log h)$.
  (T. M. Chan, 1996)
A better characterization

- Consider a walk in clockwise direction on the vertices of a closed polygon.
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The incremental paradigm

![Diagram showing upper and lower hulls of a polygon with vertices labeled p1 to pn.](image)
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- Insert points in P one by one and update the solution at each step.
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- Insert points in $P$ one by one and update the solution at each step.
- We compute the upper hull first. The upper hull contains the convex hull edges that bound the convex hull from above.
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The incremental paradigm

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- Sort the points in $P$ from left to right.
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Sort $P$ according to $x$-coordinate to generate
a sequence of points $p[1], p[2], \ldots, p[n]$;
A Naive Algorithm

Input: A set P of n points in the plane
Output: Convex Hull of P
Sort P according to x-coordinate to generate
    a sequence of points p[1], p[2], ..., p[n];
Insert p[1] and then p[2] in a list L_U;

```plaintext
for i = 3 to n {
    Append p[i] to L_U;
    while (L_U contains more than two points AND
        the last three points in L_U do not make a right turn) {
        Delete the middle of the last
        three points from L_U;
    }
}
```
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The Algorithm in Action

\[ p_1, \ldots, p_n \]
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\[ p_1 \quad p_n \]
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$\{p_1, \ldots, p_n\}$
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$p_1 \rightarrow \ldots \rightarrow p_n$
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- Hence, the total time complexity is $O(n \log n)$. 
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Can we find, as a function of \( n \), the number of cameras that suffices to guard \( P \)?
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Art Gallery Problem

Can the bound be reduced?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $P$. 

We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.

Choose the smallest color class to guard $P$.

Hence, $\lfloor \frac{n}{3} \rfloor$ guards suffice.

But, does a 3-coloring always exist?
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- Hence, $\left\lceil \frac{n}{3} \right\rceil$ guards suffice.
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Art Gallery Problem

A 3-coloring always exist

Consider the dual graph $G_T$ of $P$. $G_T$ is a tree as $P$ has no holes. Do a DFS on $G_T$ to obtain the coloring. Place guards at those vertices that have color of the minimum color class. Hence, $\lceil n/3 \rceil$ guards are sufficient to guard $P$.

Necessity? Are $\lceil n/3 \rceil$ guards sometimes necessary?
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Art Gallery Theorem

Final Result

For a simple polygon with $n$ vertices, $\left\lfloor \frac{n}{3} \right\rfloor$ cameras are always sufficient and occasionally necessary to have every point in the polygon visible from at least one of the cameras.
References

References II

- http://www.algorithmic-solutions.com
- http://www.cgal.org
Thank you!