Algorithm design in Perfect Graphs

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What is it to be Perfect?

- Introduced by Claude Berge in early 1960s
- Coloring number and clique number are one and the same for all induced subgraphs of a Graph
- Note that the coloring number is at least the clique number
- Are the even unequal? - Odd cycles!!!
- To be perfect, induced subgraphs cannot be odd cycles

Exercise in Coloring

- For any given two integers, o and c, does there exist a graph whose coloring number is c and clique number is o.
- For o=2 and c=3, answer is obviously yes.
- Construct a graph for o=2 and c=4.
- Answered by Lovasz for arbitrary values of o and c.
- Check text on Graph Theory by Bondy and Murty.

Perfect Questions

- Is a given graph Perfect?
- Is there a characterization of perfect graphs?
- Is a graph minimally imperfect?
- Do any hard computational exercises become easy on these graphs?
- Are there interesting sub-classes?

This talk: A survey of the first 4 and a sample of the last question

Characterizations

- Strong Perfect Graph Theorem

  A Graph is perfect if and only if it does not contain a odd cycle or its complement as an induced subgraph.
- Conjectured by Berge in 1960
- A forbidden subgraph characterization.
- Conjecture settled after many years of research in the first decade of this century.
- Come up with a verification algorithm?

Results along the way

- Weak Perfect Graph Theorem [Lovasz, Fulkerson]

  A Graph is perfect if and only if its complement is perfect.

  Further, G is perfect if and only if for each induced subgraph H, the alpha-omega product is at least the number of vertices in H.

- Consequently, independence number is same as clique cover number for all induced subgraph of a perfect graph.

Polyhdedral Combinatorics

- Main goal-understanding the geometric structure of a solution space.

  Visualize the convex hull and find a system of inequalities that specify exactly the convex hull
- Consider the convex hull of stable set incidence vectors
- Consider the clique inequalities
- G is perfect if and only if the convex hull and clique inequality polytope are identical

Summary of Survey

- Perfect graphs are motivated by coloring issues.
- Connects combinatorial understanding to polyhedral structure in a very rich and fundamental way

Geometric Algorithms and Combinatorial Optimization – Groetschel, Lovasz, Schrijver

Algorithmic Graph Theory and Perfect Graphs – Golumbic

The Sandwich Theorem – Knuth

Interval Graphs

- A subclass of perfect graphs
- Motivated by many applications
  - Temporal reasoning issues like register allocation
- Given a set of intervals, consider the natural intersection graph for which there is one vertex per interval and an edge indicates a non-empty intersection
- Examples of interval graphs and non interval graphs

Interval Graphs are perfect

- Given a graph, find an interval representation
- Visualize the intervals as time intervals
- Color the intervals in increasing order of time
- Reuse a color whenever possible and use a new color greedily
- This proves that interval graphs are perfect.
- Key issues: given a graph, does it have an interval represenation.

Forbidden subgraphs

- Induced cycles of length more than 3
- Asteroidal triples

  3 vertices x, y, z form an asteroidal triple if for all ordering of them, there is a path from the first to third which avoids the neighbors of the second.
- Gives a polynomial time algorithm
  - Check no four form an induced cycle
  - Check no 3 form an asteroidal triple

The interval representation

- Graph is an interval graph if and only if its maximal cliques can be linearly ordered such that the set of maximal cliques containing a vertex occur consecutively in the order.
- Note that this consecutive ordering gives the interval representation
  - For each vertex, the interval associated is the interval of indices of maximal cliques that contain it
- Finding the maximal cliques and ordering them!!

Finding the maximal cliques

- Based on a structural property of graphs that do not have induced 4 cycles.
- Such graphs are called chordal/triangulated.
- Have two very special properties
  - Each minimal vertex separator is a clique
  - There are two non-adjacent vertices whose neighborhood is a clique.
- There is a perfect elimination ordering
  - Order of vertices, such that higher nbrs form a clique

Where are the maximal cliques

- Each maximal clique is in any perfect elimination ordering
- Can be constructed in polynomial time
  - Select a simplicial vertex
  - Remove it from the graph, resulting graph is still chordal
  - Iterate: the order in which the vertices where output is a perfect elimination ordering
- All maximal cliques are now found from this ordering- each maximal clique has a smallest number vertex in the order.

The Interval Assignment

- Need to order the maximal cliques linearly, such that for each vertex, those cliques containing the vertex occur consecutively in the linear order
- For each v, consider the set M(v) containing the maximal cliques containing v.
- Identify a corner vertex v.
  - There must be one if there is an interval ordering
  - If the set of intersections with M(u) not contained in M(v) form linear order under the containment partial order.

Completing the Assignment

- If a corner not found, report not an interval graph and exit.
- Else, assign the left most interval to the set
- Iteratively, select a set that is not contained in some already processed set, but intersects with it, and assign it an interval in a unique way.
- If the interval cannot be assigned consistent with the others, meaning check the intersection cardinality, report failure and exit.

Other results

- Chordal graphs are also perfect
  - By a greedy coloring using the perfect elimination ordering
- Vertex cover, maximum clique and independent set be computed in polynomial time on perfect graphs.
- Many more applications in Computational Geometry by way of visibility graphs etc.