Extremal Graph Theory

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Basic Question

- Let H be a fixed graph.
- What is the maximum number of edges in a graph G with n vertices that does not contain H as a subgraph?
- This number is denoted $ex(n,H)$.
- A graph G with n vertices and $ex(n,H)$ edges that does not contain H is called an extremal graph for H.

Mantel's Theorem (1907)

$$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

- The only extremal graph for a triangle is the complete bipartite graph with parts of nearly equal sizes.

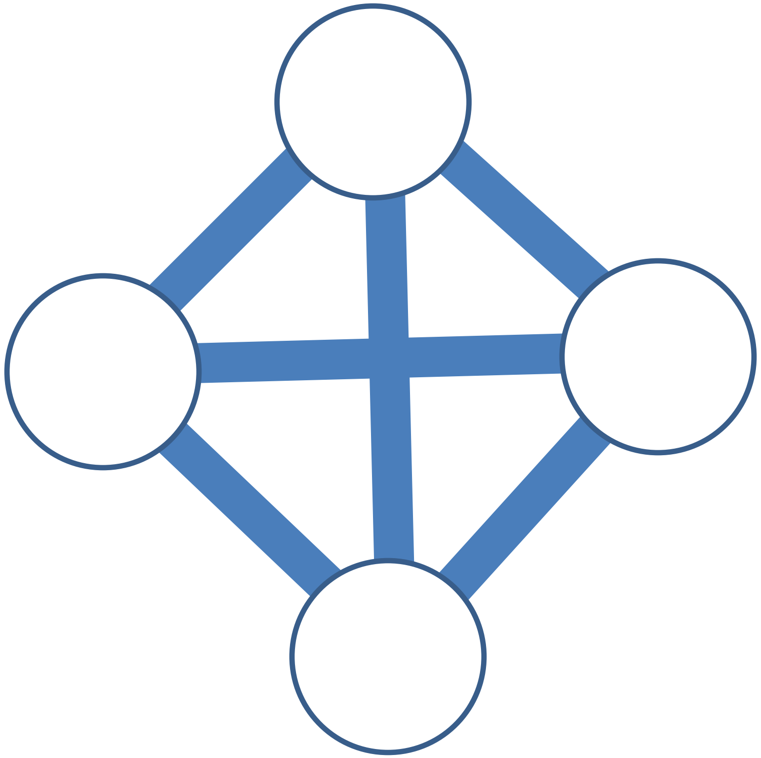
Complete Bipartite graph

Turan’s theorem (1941)

$$ex(n, K_t) \leq \left(\frac{t-2}{2(t-1)} \right) n^2$$

- Equality holds when *n* is a multiple of *t*-1.
- The only extremal graph is the complete (*t*-1)-partite graph with parts of nearly equal sizes.

Complete Multipartite Graph



Proofs of Turan’s theorem

- Many different proofs.
- Use different techniques.
- Techniques useful in proving other results.
- Algorithmic applications.
- “BOOK” proofs.

Induction

- The result is trivial if $n <= t$-1.
- Suppose $n >= t$ and consider a graph G with maximum number of edges and no K_t.
- G must contain a K_{t-1}.
- Delete all vertices in K_{t-1}.
- The remaining graph contains at most $\left(\frac{t-2}{2(t-1)}\right)(n-t+1)^2$ edges.

Induction

- No vertex outside K_{t-1} can be joined to all vertices of K_{t-1}.
- Total number of edges is at most

$$\left(\frac{t-2}{2(t-1)}\right)(n-t+1)^2 + \frac{(t-1)(t-2)}{2} +$$

$$(n-t+1)(t-2) = \left(\frac{t-2}{2(t-1)}\right)n^2$$

Greedy algorithm

- Consider any extremal graph and let *v* be a vertex with maximum degree Δ.
- The number of edges in the subgraph induced by the neighbors of v is at most

$$\left(\frac{t-3}{2(t-2)}\right)\Delta^2$$

- Total number of edges is at most

$$\Delta(n-\Delta)+\left(\frac{t-3}{2(t-2)}\right)\Delta^2$$

Greedy algorithm

- This is maximized when

$$\Delta = \left(\frac{t-2}{t-1} \right) n$$

- The maximum value for this Δ is

$$\left(\frac{t-2}{2(t-1)} \right) n^2$$

Another Greedy Algorithm

- Consider any graph that does not contain K_t.
- Duplicating a vertex cannot create a K_t.
- If the graph is not a complete multipartite graph, we can increase the number of edges without creating a K_t.
- A graph is multipartite if and only if non-adjacency is an equivalence relation.

Another Greedy Algorithm

- Suppose *u, v, w* are distinct vertices such that *vw* is an edge but *u* is not adjacent to both *v* and *w*.
- If degree(*u*) < degree (*v*), duplicating *v* and deleting *u* increases number of edges, without creating a K_t.
- Same holds if degree(*u*) < degree(*w*).
- If degree(*u*) >= degree(*v*) and degree(*w*), then duplicate *u* twice and delete *v* and *w*.

Another Greedy Algorithm

- So the graph with maximum number of edges and not containing K_t must be a complete multipartite graph.
- Amongst all such graphs, the complete (t-1)-partite graph with nearly equal part sizes has the maximum number of edges.
- This is the only extremal graph.

Erdős-Stone Theorem

- What can one say about *ex(n,H)* for other graphs *H*?
- Observation:

$$ex(n,H) \geq ex(n,K_{\chi(H)})$$

- χ (*H*) is the chromatic number of *H*.
- This is almost exact if χ (*H*) >= 3.

Erdős-Stone Theorem

- For any ε > 0 and any graph *H* with χ (*H*) >= 3 there exists an integer n_0 such that

  for all $n >= n_0$

$$ex(n, H) \leq (1+\varepsilon) ex(n, K_{\chi(H)})$$

- What about bipartite graphs (χ (*H*) = 2)?
- Much less is known.

Four Cycle

$$ex(n, C_4) = \theta(n^{3/2})$$

- For all non-bipartite graphs *H*

$$ex(n, H) = \Omega(n^2)$$

Four Cycle

- Consider the number of paths (*u*,*v*,*w*) of length two.
- The number of such paths is $\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2}$
- d_i is the degree of vertex *i*.
- The number of such paths can be at most

$$\frac{(n-1)(n-2)}{2}$$

- No two paths can have the same pair of endpoints.

Four Cycle

If $$\sum_{i=1}^{n} d_i = 2m$$

then $$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2} = \Omega\left(\frac{m^2}{n}\right)$$

which implies the result.

Matching

- A matching is a collection of disjoint edges.
- If *M* is a matching of size *k* then

$$ex(n,M)=\max\left(\binom{2k-1}{2},\binom{k-1}{2}+(n-k+1)(k-1)\right)$$

- Extremal graphs are K_{2k-1} or $K_{k-1}+E_{n-k+1}$

Path

- If *P* is a path with *k* edges then

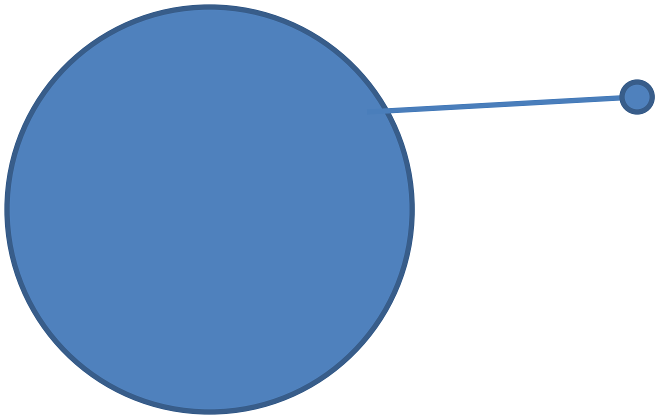
$$ex(n,P) <= \left(\frac{k-1}{2}\right)n$$

- Equality holds when *n* is a multiple of *k*.
- Extremal graph is mK_k.
- **Erdős-Sós Conjecture :** same result holds for any tree *T* with *k* edges.

Hamilton Cycle

- Every graph *G* with *n* vertices and more than $\frac{(n-1)(n-2)}{2}+1$ edges contains a Hamilton cycle.

- The only extremal graph is a clique of size *n-1* and 1 more edge.

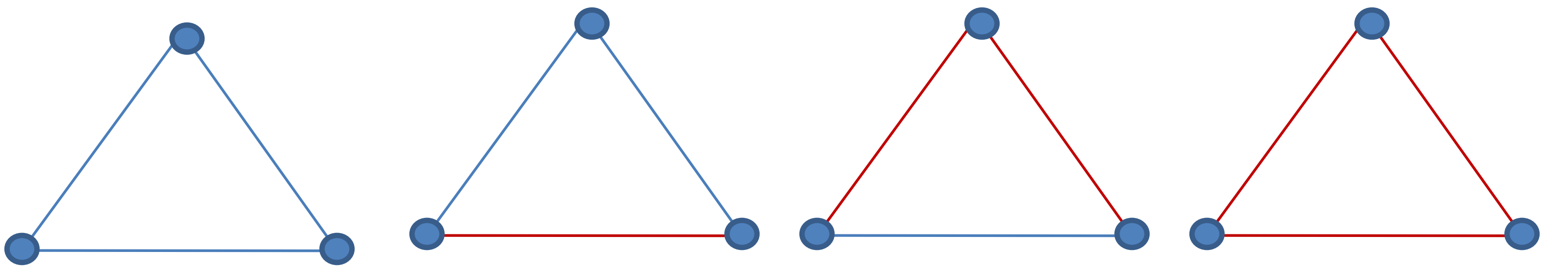


Colored Edges

- Extremal graph theory for edge-colored graphs.
- Suppose edges have an associated color.
- Edges of different color can be parallel to each other (join same pair of vertices).
- Edges of the same color form a simple graph.
- Maximize the number of edges of each color avoiding a given colored subgraph.

Colored Triangles

- Suppose there are two colors , red and blue.



- What is the largest number *m* such that there exists an *n* vertex graph with *m* red and *m* blue edges, that does not contain a specified colored triangle?

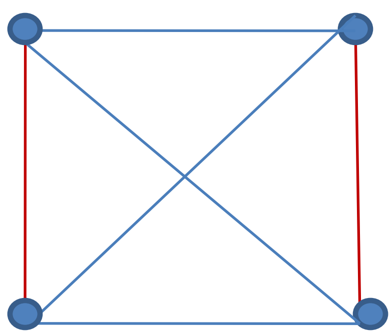
Colored Triangles

- If both red and blue graphs are complete bipartite with the same vertex partition, then no colored triangle exists.
- More than $\left\lfloor \frac{n^2}{4} \right\rfloor$ red and blue edges required.

- Also turns out to be sufficient to ensure existence of **all** colored triangles.

Colored 4-Cliques

- By the same argument, more than $n^2/3$ red and blue edges are required.
- However, this is not sufficient.
- Different extremal graphs depending upon the coloring of K_4.



Colored 4-Cliques

- Red clique of size n/2 and a disjoint blue clique of size n/2.
- Vertices in different cliques joined by red and blue edges.
- Number of red and blue edges is $\approx \frac{3n^2}{8}$

General Case

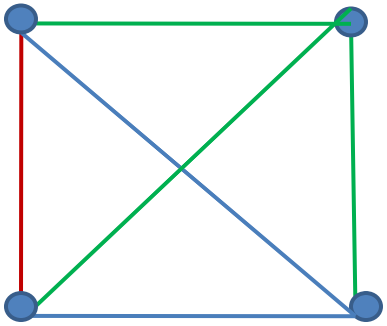
- Such colorings, for which the number of edges required is more than the Turan bound exist for k = 4, 6, 8.
- We do not know any others.
- **Conjecture:** In all other cases, the Turan bound is sufficient!
- Proved it for k = 3 and 5.

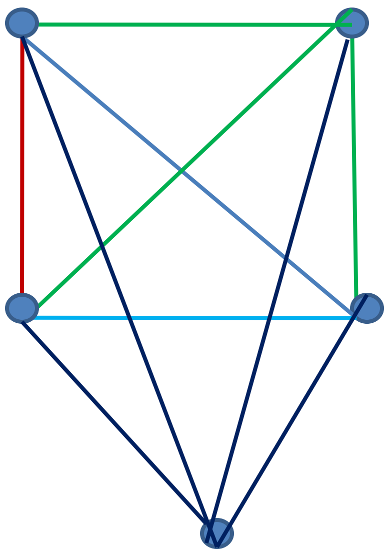
Colored Turan's Theorem

- Instead of requiring *m* edges of each color, only require that the total number of edges is *cm*, where *c* is the number of colors.
- How large should *m* be to ensure existence of a particular colored k-clique?
- For what colorings is the Turan bound sufficient?

Star-Colorings

- Consider an edge-coloring of K_k with k-1 colors such that edges of color i form a star with i edges. (call it a star-coloring)





Conjecture

- Suppose *G* is a multigraph with edges of *k-1* different colors and total number of edges is more than .
- *G* contains every star-colored K_k . Verified only for *k <= 4*.
- Extremal graphs can be obtained by replicating edges in the Turan graph.
- Other extremal graphs exist.

Colored Matchings

- *G* is a *c* edge-colored multigraph with *n* vertices and number of edges of each color is more than

$$ex(n,M) = \max\left(\binom{2k-1}{2}, \binom{k-1}{2} + (n-k+1)(k-1)\right)$$

- *G* contains every *c* edge-colored matching of size *k*.
- Proved for *c = 2* and for all *c* if *n >= 3k*.

Colored Hamilton Cycles

- *G* is *c* edge-colored multigraph with *n* vertices and more than $\frac{(n-1)(n-2)}{2}+1$ edges of each color.
- G contains all possible *c* edge-colored Hamilton cycles.
- Proved for *c* <= 2, and for *c = 3* and *n* sufficiently large.

References

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Thank You