Introduction to Computational Geometry

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Outline

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- There are many areas in computer science like computer graphics, computer vision and image processing, robotics, computer-aided designing (CAD), geographic information systems (GIS), etc. that give rise to geometric problems.
- If one assumes Michael Ian Shamos's thesis [Shamos M. I., 1978] as the starting point, then this branch of study is around thirty five years old.

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- CG algorithms suffer from the curse of degeneracies. So, we would make certain simplifying assumptions at times like no three points are collinear, no four points are cocircular, etc.
- Programming in CG is a little difficult. Fortunately, libraries like LEDA [LEDA, www.algorithmic-solutions.com] and CGAL [CGAL, www.cgal.com] are now available. These libraries implement various data structures and algorithms specific to CG.

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- First we consider some geometric primitives, that is, problems that arise frequently in computational geometry.
- Then we study a few classical CG problems.

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Area Computation

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Given a simple polygon P of n vertices, compute its area.

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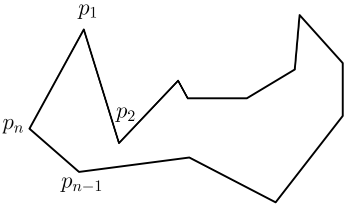
Definition

A sinple polygon is the region of a plane bounded by a finite collection of line segments forming a simple closed curve.



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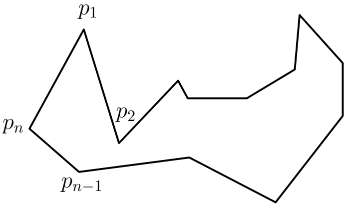
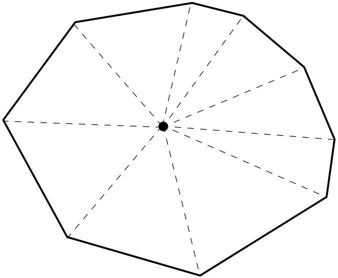
- Let us first solve the problem for convex polygon.



Area Computation

Area of a convex polygon

Find a point inside P, draw n triangles and compute the area.



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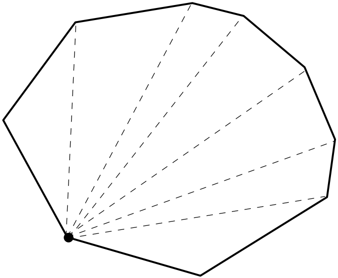
A better idea for convex polygon

We can triangulate P by non-crossing diagonals into $n-2$ triangles and then find the area.



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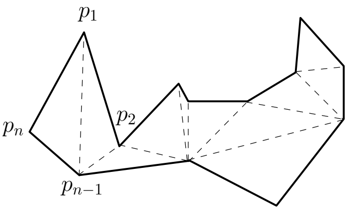
Area of a simple polygon

We can do likewise.



Area Computation

Result

If P be a simple polygon with n vertices with coordinates of the vertex p_i being (x_i, y_i), $1 \leq i \leq n$, then twice the area of P is given by

$$2\mathcal{A}(P) = \sum_{i=1}^{n}(x_i y_{i+1} - y_i x_{i+1})$$

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Time complexity

We can triangulate P by a very complicated $O(n)$ time algorithm [Chazelle B., 1991] OR by a *more or less simple* $O(n \log n)$ time algorithm [Berg M. d. et. al., 1997].

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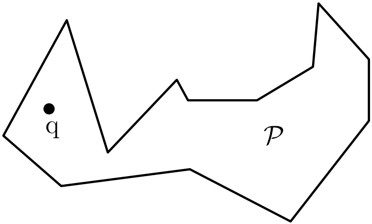
Given a simple polygon P of n points, and a query point q, is $q \in P$?



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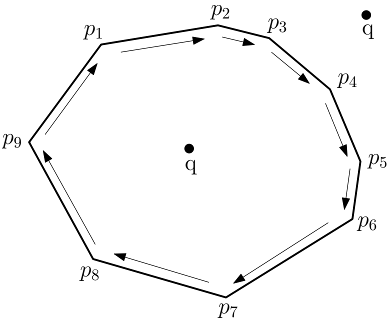

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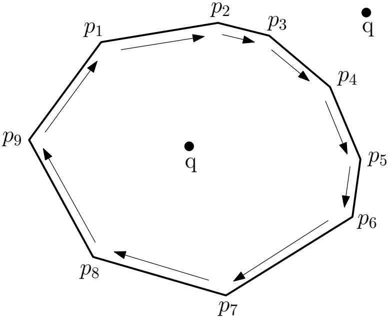
- Can be done in $O(n)$.
- Takes a little effort to do it in $O(\log n)$. Left as an exercise.



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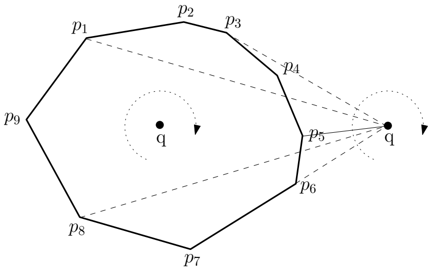
Point Inclusion

Another idea for convex polygon

Stand at q and walk around the polygon.



Total angular turn around q is 2π if $q \in \mathcal{P}$, else, 0

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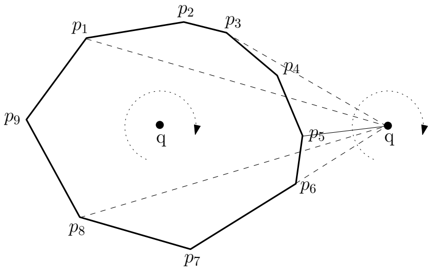
Point inclusion for polygon

We can show that the same result holds for a simple polygon also.



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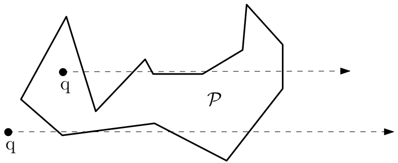
Still another technique: Ray Shooting

- Shoot a ray and count the number of crossings with edges of P. If it is odd, then $q \in P$. If it is even, then $q \notin P$.



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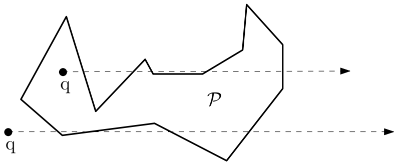

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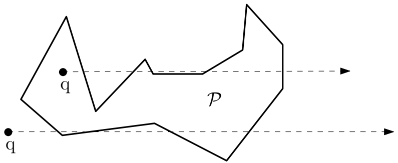
Still another technique: Ray Shooting

- Shoot a ray and count the number of crossings with edges of P. If it is odd, then $q \in P$. If it is even, then $q \notin P$.
- Time complexity is $O(n)$.
- Some degenerate cases need to be taken care of.



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A set $\mathcal{S} \subset \mathcal{R}^2$ is convex if for any two points $p, q \in \mathcal{S}$, $\overline{pq} \in \mathcal{S}$.

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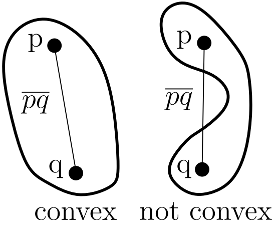

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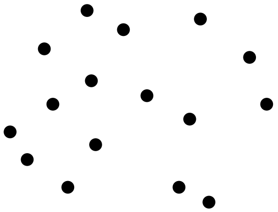

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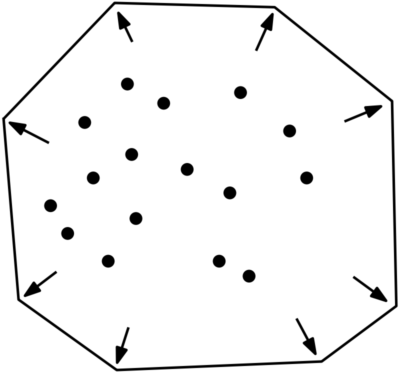

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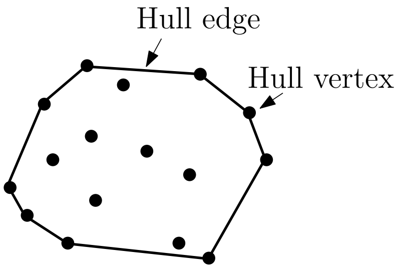

Convex Hull Problem

Problem

Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$.

A Naive Algorithm

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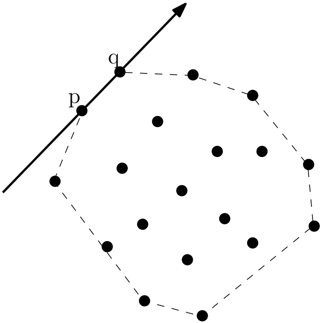

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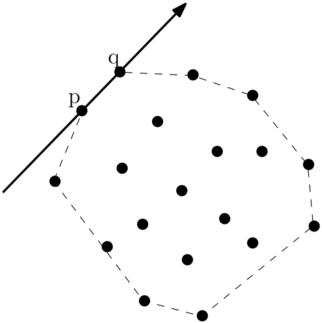
- If a line segment has all the other $n - 2$ points on one side of it, then it is a hull edge.
- We need $\binom{n}{2}(n-2) = O(n^3)$ time.



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- Better characterizations lead to better algorithms.
- How much better can we make?
- Leads to the notion of lower bound of a problem.
- The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$.

Optimal Algorithms

- Grahams scan, time complexity $O(nlogn)$ (Graham, R.L., 1972).
- Divide and conquer algorithm, time complexity $O(nlogn)$ (Preparata, F. P. and Hong, S. J., 1977).
- Jarvis's march or gift wrapping algorithm, time complexity $O(nh)$ where h is the number of vertices of the convex hull. (Jarvis, R. A., 1973)
- Most efficient algorithm to date is based on the idea of Jarvis's march, time complexity $O(nlogh)$ (T. M. Chan, 1996).

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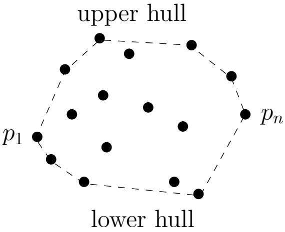
A better characterization

- Consider a walk in clockwise direction on the vertices of a closed polygon.



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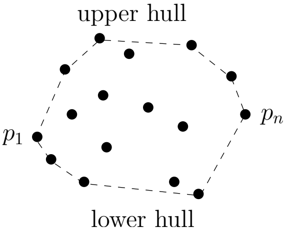

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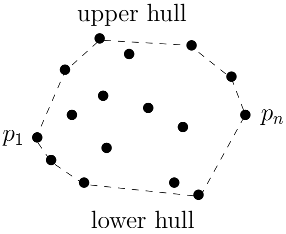

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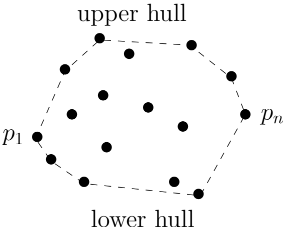

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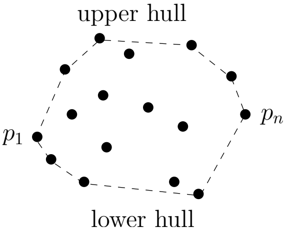
- We compute the upper hull first. The upper hull contains the convex hull edges that bound the convex hull from above.



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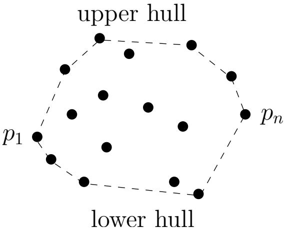
- The lower hull can be computed in a similar fashion.



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Sort P according to x-coordinate to generate
   a sequence of points p[1], p[2], ..., p[n];
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   Append p[i] to L_U;
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         Delete the middle of the last
         three points from L_U;
   }
}
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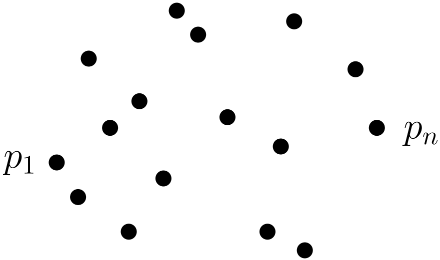
The Algorithm in Action



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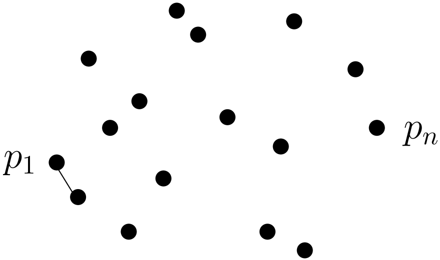

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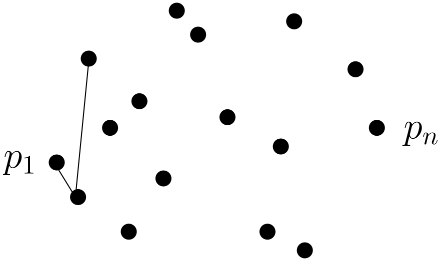

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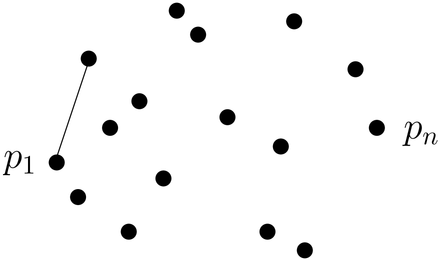

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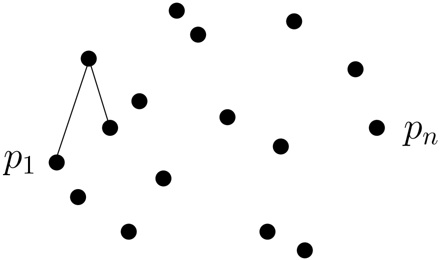

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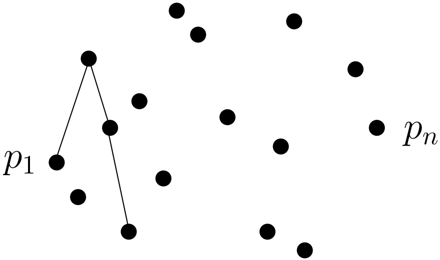

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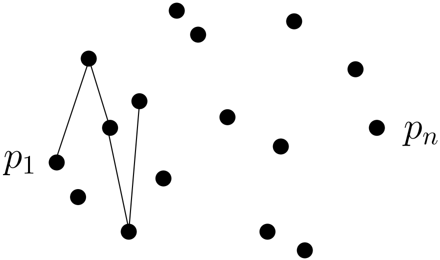

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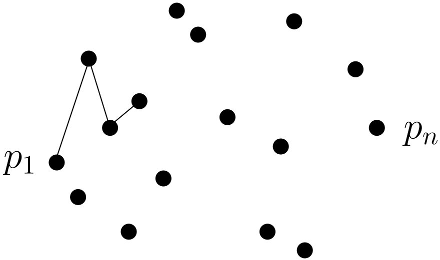

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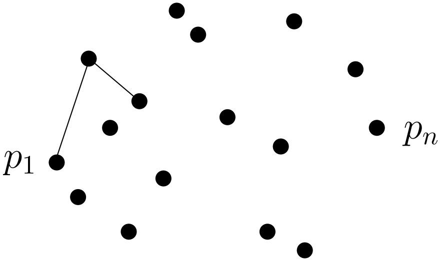

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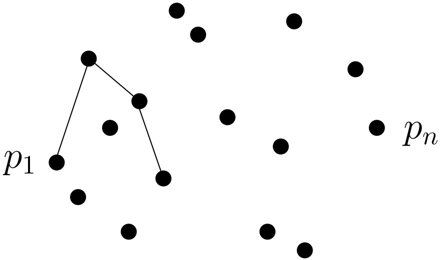

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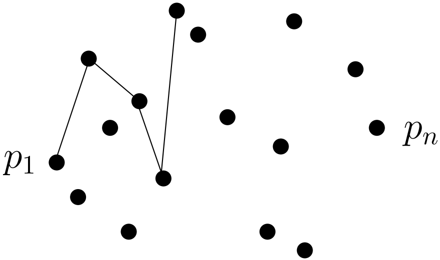

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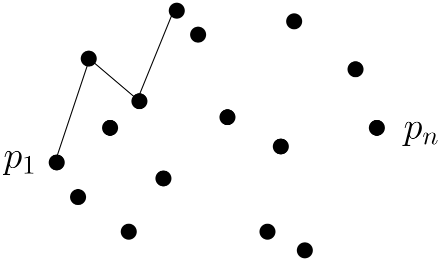

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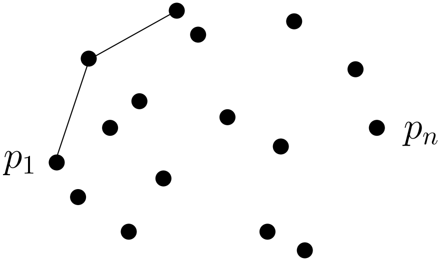

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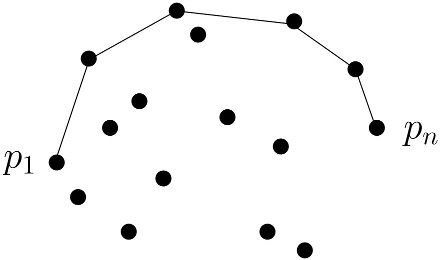

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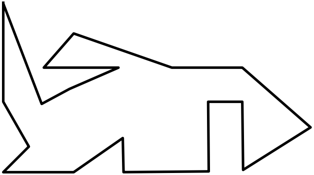
Outline

Art Gallery Problem

The problem

Given a simple polygon $\mathcal{P}$ of n vertices, find the minimum number of cameras that can guard $\mathcal{P}$.



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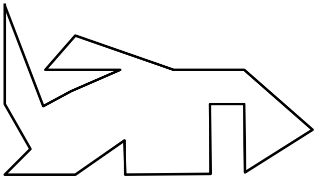

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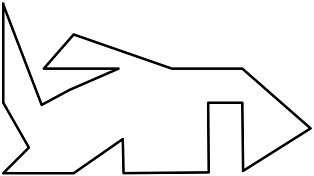

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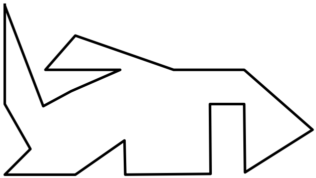

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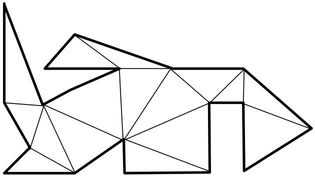
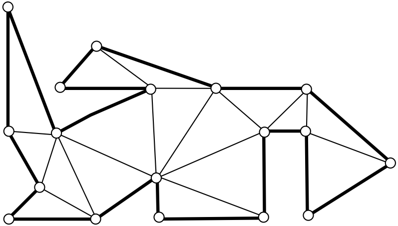
- Recall $\mathcal{P}$ can be triangulated into $n-2$ triangles. Place a guard in each triangle.



Art Gallery Problem

Can the bound be reduced?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $\mathcal{P}$.



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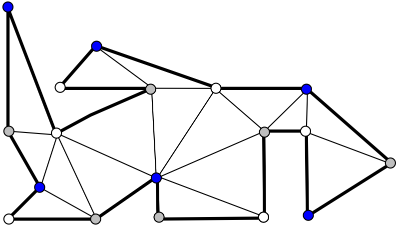
Can the bound be reduced?

- Place guards at vertices of the triangulation $\mathcal{T}$ of $\mathcal{P}$.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.



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- Place guards at vertices of the triangulation $\mathcal{T}$ of $\mathcal{P}$.
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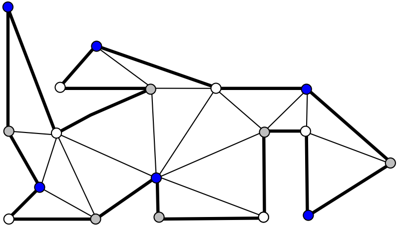

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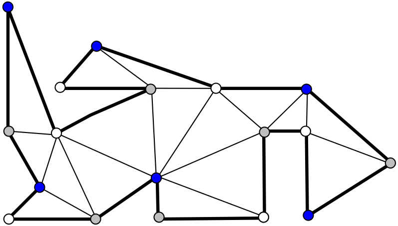
- Hence, $\lfloor \frac{n}{3} \rfloor$ guards suffice.



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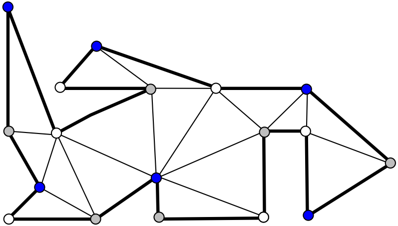
- But, does a 3-coloring always exist?



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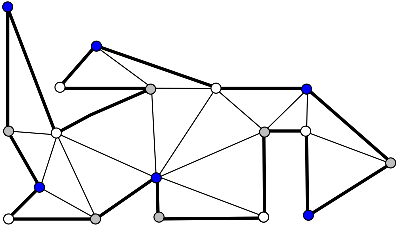

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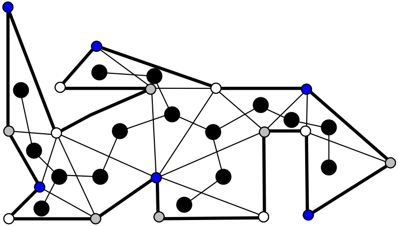
- Consider the dual graph $\mathcal{G}_\mathcal{T}$ of $\mathcal{T}$ of $\mathcal{P}$.



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- Consider the dual graph $\mathcal{G}_\mathcal{T}$ of $\mathcal{T}$ of $\mathcal{P}$.
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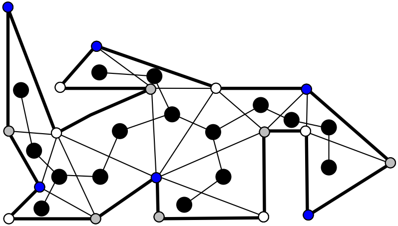

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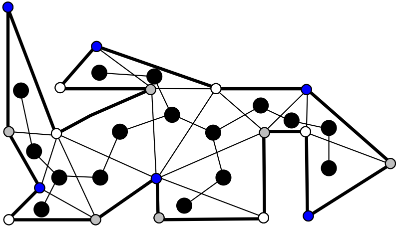

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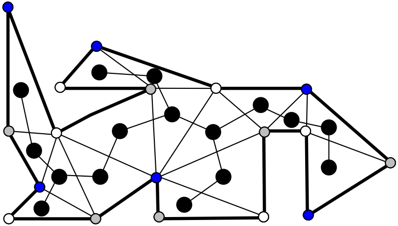
- Do a DFS on $\mathcal{G}_{\mathcal{T}}$ to obtain the coloring.
- Place guards at those vertices that have color of the minimum color class. Hence, $\lfloor \frac{n}{3} \rfloor$ guards are sufficient to guard $\mathcal{P}$.



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Necessity?

Are $\lfloor \frac{n}{3} \rfloor$ guards sometimes necessary?

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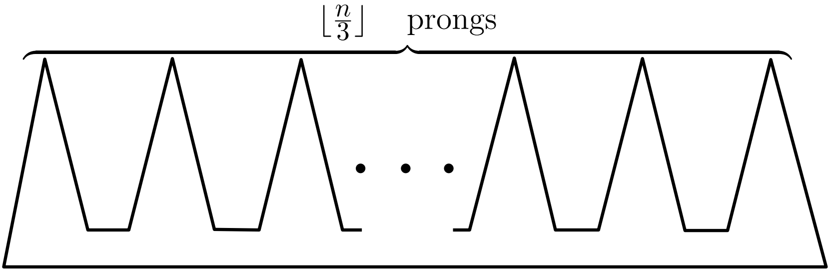

Art Gallery Theorem

Final Result

For a simple polygon with n vertices, $\lfloor \frac{n}{3} \rfloor$ cameras are always sufficient and occasionally necessary to have every point in the polygon visible from at least one of the cameras.

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Thank you!