Introduction to Combinatorial Geometry

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Helly's Theorem

Theorem

Let C be a collection of convex objects in R^d. If every $d + 1$ objects in C have a common intersection, then all the objects in C have a common intersection.

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  - **Extremal proof** [Mustafa and Ray, 2007]

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  - Proof generalizes to d dimensions.

Intervals in 1D

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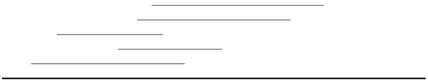
- Every 2 intervals in S intersect



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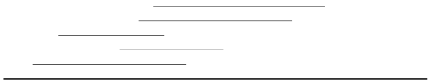

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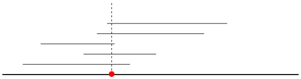

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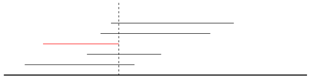

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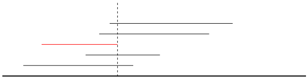

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- Construct a point p that is contained in all the intervals
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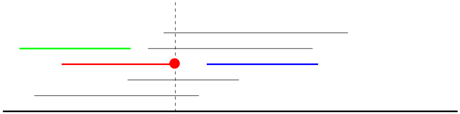
- Proof by contradiction



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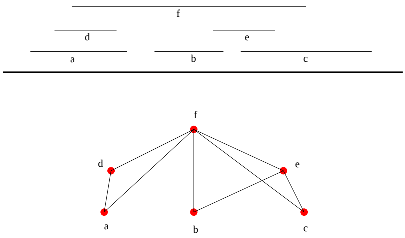

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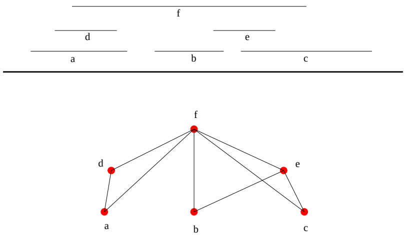

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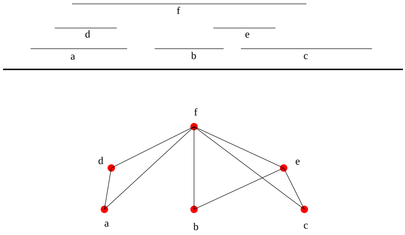

- V - set of intervals s_i
- $(s_i, s_j) \in E$ if intervals s_i and s_j intersect

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  - Greedy Algorithm to solve the problem (Exercise) with **Proof of correctness**
  - Extension: What if jobs have different profits? (Use dynamic programming)

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  - Is it true?

Circles in 2D

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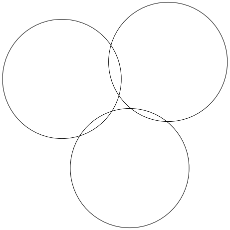

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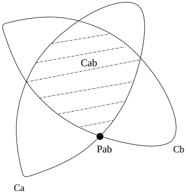

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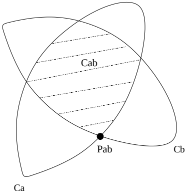

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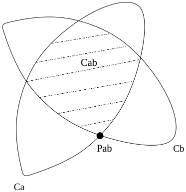

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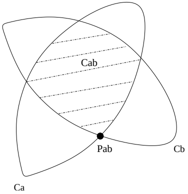

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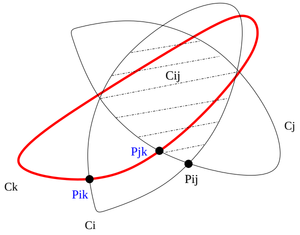

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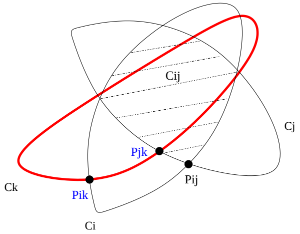

- If p_{ij} is not contained in C_k

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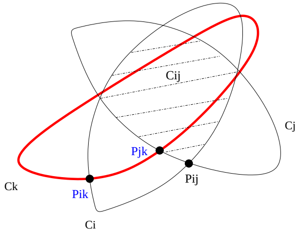

- If p_{ij} is not contained in C_k
  - p_{jk} higher than p_{ij} - Contradiction

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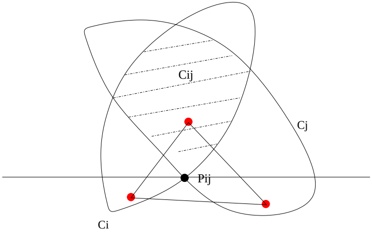

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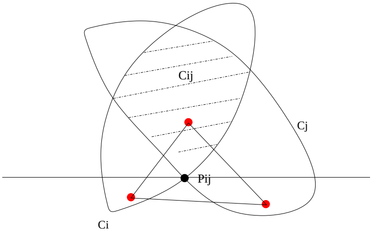

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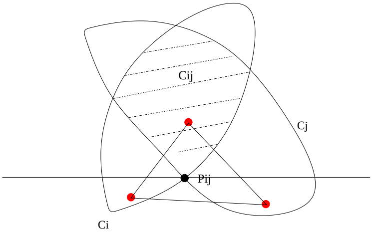

- $C_{ij} \cap C_k \neq \emptyset$ (Every 3 objects intersect)
- C_k intersect both C_i and C_j below p_{ij}
  - p_{ik} and p_{jk} must be lower than p_{ij}

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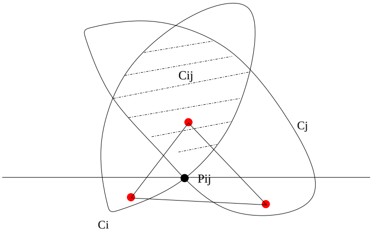

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  - p_{ik} and p_{jk} must be lower than p_{ij}
- By convexity, p_{ij} is contained in C_k

Hadwiger-Debrunner (p, q) problem

Definition

For any positive integers, p, q, let C be a family of convex objects C in $\mathbb{R}^d$ with $[p, q]$-property. How many points are needed to pierce C?

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- Constant number of points pierce all objects (Weak ϵ-nets)

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Let P be a set of n points in the plane. There exists a point p in the plane that is contained in every convex object containing $> \frac{2}{3}n$ points of P.

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- Applying Helly theorem, there exists a point p contained in all such convex objects
- The constant $\frac{2}{3}$ is the best possible

Strong Centerpoint

- Can we restrict the centerpoint to belong to P?

Strong Centerpoint

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  - NO
  - No, even for halfspaces

Strong Centerpoint for axis parallel rectangles

Theorem (Strong Centerpoint Theorem (Ashok, Azmi, G. '14))

Let P be a set of n points in the plane. There exists a point $p \in P$ that is contained in every rectangle containing $> \frac{3}{4}n$ points of P.

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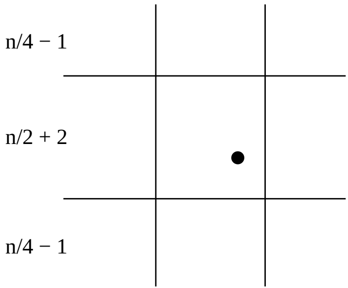
Theorem (Strong Centerpoint Theorem (Ashok, Azmi, G. '14))

Let P be a set of n points in the plane. There exists a point $p \in P$ that is contained in every rectangle containing $> \frac{3}{4}n$ points of P.

- The constant $\frac{3}{4}$ is the best possible

Axis-Parallel Rectangles



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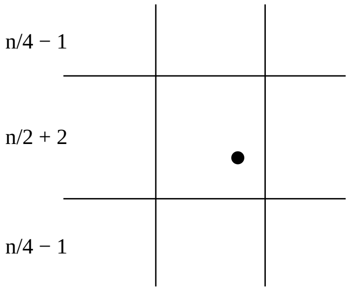

The second column contains $\frac{n}{2} + 2$ points.

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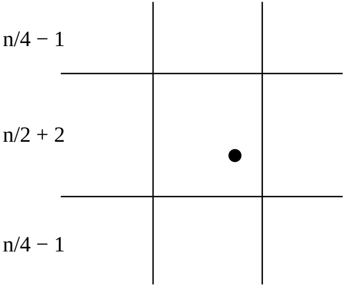
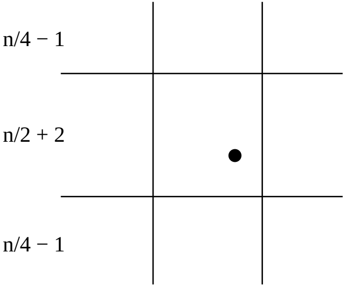


The second column contains $\frac{n}{2} + 2$ points.

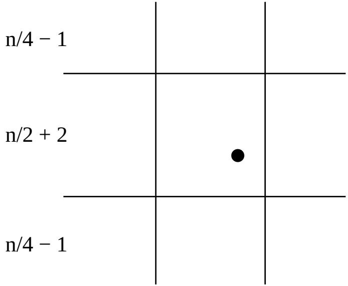
Since regions (1,2) and (3,2) contain at most $\frac{n}{4} - 1$ points each, the region (2,2) is not empty

Axis-Parallel Rectangles



Select any point from region (2,2) as the ϵ-net.

Axis-Parallel Rectangles



Select any point from region (2,2) as the ϵ-net.

Any axis-parallel rectangle that does not contain the chosen point will have $\leq \frac{3n}{4}$ points.

Small Weak Epsilon Nets

Small Weak Epsilon Nets

- Select many points instead of just one

Theorem (Generalized Centerpoints)

Let P be a set of n points in the plane. There exists a set of i points Q in the plane such that $c \cap Q \neq \emptyset$ for any convex object c containing $> \epsilon_i n$ points of P.

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- Bounds for ϵ_i?

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- Extension: $\epsilon_2 = 4/7$

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	Rectangles		Halfspaces		Disks		Convex sets	
	LB	UB	LB	UB	LB	UB	LB	UB
ϵ_1	1/2		2/3		2/3		2/3	
ϵ_2	2/5		1/2		1/2	4/7	4/7	
ϵ_3	1/3		0		1/4	8/15	5/11	8/15

Table: Summary of bounds [Aronov et al '09, MR '07]

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Table: Summary of bounds [Aronov et al '09, MR '07]

- Open problem: Find exact value of ϵ_i for small i?

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ϵ_2	5/9	5/8	3/5	2/3	3/5	2/3
ϵ_3	9/20	5/9	1/2		1/2	2/3

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Small Strong epsilon nets

- Restrict $Q \subseteq P$

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Table: Summary of bounds [AAG '10]

- Open problem: Find exact value (for $k = 2$)

First Selection Lemma (FSL)

- For induced triangles in R^2, Boros and Füredi (1984), showed that the centerpoint is present in $\frac{n^3}{27}$ (constant fraction) triangles induced by P. This constant is tight.

FSL for Axis-Parallel Rectangles in R^2

Theorem (Ashok, G., Mishra, Rajgopal '13)

There exists a point p in R^2, which is present in at least $\frac{n^2}{8}$ axis-parallel rectangles induced by P. This bound is tight.

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There exists a point p in R^2*, which is present in at least* $\frac{n^2}{8}$ *axis-parallel rectangles induced by* P*. This bound is tight.*

The tightness of the bound - P distributed around the boundary of a circle.

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Theorem (Ashok, G., Mishra, Rajgopal '13)

There exists a point $p \in P$ such that p is contained in at least $\frac{n^2}{16}$ induced rectangles. This bound is tight.

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Open problem: FSL for boxes in higher dimension

FSL for Disks in R^2

Theorem (Ashok, G., Mishra, Rajgopal '13)

There exists a point p in R^2, which is present in at least $\frac{n^2}{6}$ disks induced by P.

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Open problem: Obtain tight bounds for disks

Questions?