Algorithm design in Perfect Graphs

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What is it to be Perfect?

- Introduced by Claude Berge in early 1960s
- Coloring number and clique number are one and the same for all induced subgraphs of a Graph
- Note that the coloring number is at least the clique number
- Are they even unequal? - Odd cycles!!!
- To be perfect, induced subgraphs cannot be odd cycles

Exercise in Coloring

- For any given two integers, o and c, does there exist a graph whose coloring number is c and clique number is o.
- For o=2 and c=3, answer is obviously yes.
- Construct a graph for o=2 and c=4.
- Answered by Lovasz for arbitrary values of o and c.
- Check text on Graph Theory by Bondy and Murty.

Perfect Questions

- Is a given graph Perfect?
- Is there a characterization of perfect graphs?
- Is a graph minimally imperfect?
- Do any hard computational exercises become easy on these graphs?
- Are there interesting sub-classes?

This talk: A survey of the first 4 and a sample of the last question

Characterizations

- Strong Perfect Graph Theorem

A Graph is perfect if and only if it does not contain a odd cycle or its complement as an induced subgraph- last decade Chudnovsky..

- Conjectured by Berge in 1960

- A forbidden subgraph characterization.
- Conjecture settled after many years of research in the first decade of this century.
- Come up with a verification algorithm?

Results along the way

- Weak Perfect Graph Theorem [Lovasz, Fulkerson]

A Graph is perfect if and only if its complement is perfect.

Further, G is perfect if and only if for each induced subgraph H, the alpha-omega product is at least the number of vertices in H.

- Consequently, independence number is same as clique cover number for all induced subgraph of a perfect graph.

Polyhdedral Combinatorics

- Main goal-understanding the geometric structure of a solution space.

Visualize the convex hull and find a system of inequalities that specify exactly the convex hull

- Consider the convex hull of stable set incidence vectors
- Consider the clique inequalities
- G is perfect if and only if the convex hull and clique inequality polytope are identical

Summary of Survey

- Perfect graphs are motivated by coloring issues.
- Connects combinatorial understanding to polyhedral structure in a very rich and fundamental way

Geometric Algorithms and Combinatorial Optimization – Groetschel, Lovasz, Schrijver

Algorithmic Graph Theory and Perfect Graphs – Golumbic

The Sandwich Theorem – Knuth

Interval Graphs

- A subclass of perfect graphs
- Motivated by many applications
  - Temporal reasoning issues like register allocation
- Given a set of intervals, consider the natural intersection graph for which there is one vertex per interval and an edge indicates a non-empty intersection
- Examples of interval graphs and non interval graphs

Interval Graphs are perfect

- Given a graph, find an interval representation
- Visualize the intervals as time intervals
- Color the intervals in increasing order of time
- Reuse a color whenever possible and use a new color greedily
- This proves that interval graphs are perfect.
- Key issues: given a graph, does it have an interval represenation.

Forbidden subgraphs

- Induced cycles of length more than 3
- Asteroidal triples

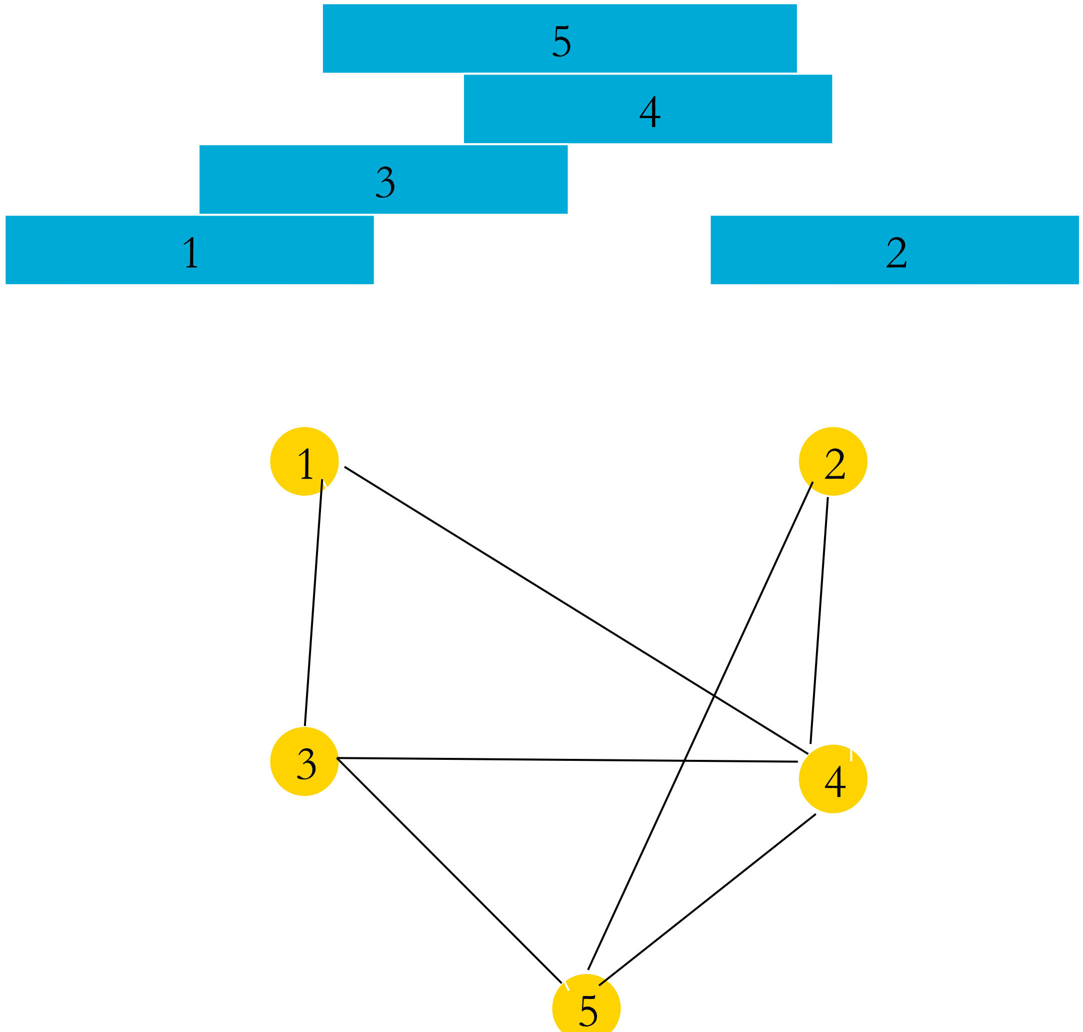
3 vertices x, y, z form an asteroidal triple if for all ordering of them, there is a path from the first to third which avoids the neighbors of the second.

- Gives a polynomial time algorithm
  - Check no four form an induced cycle
  - Check no 3 form an asteroidal triple

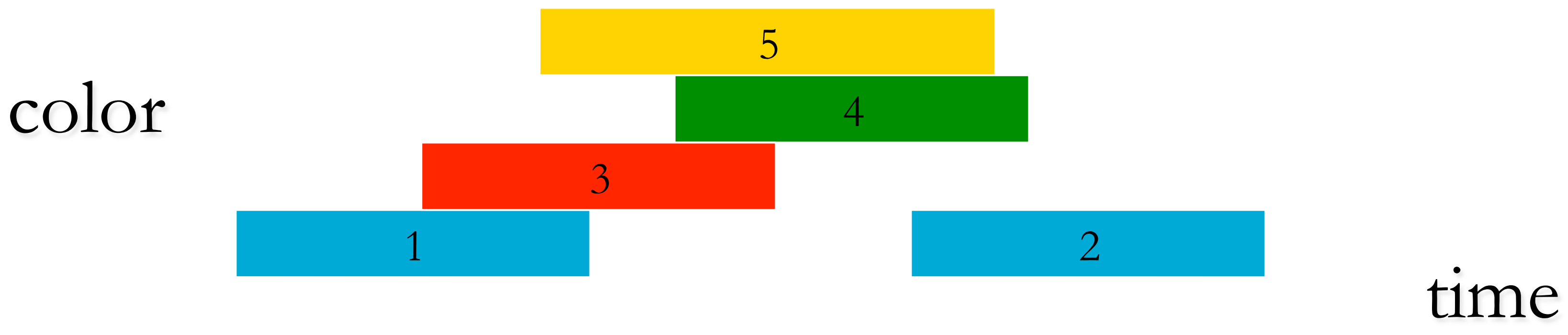
The interval representation

- Graph is an interval graph if and only if its maximal cliques can be linearly ordered such that the set of maximal cliques containing a vertex occur consecutively in the order.
- Note that this consecutive ordering gives the interval representation
  - For each vertex, the interval associated is the interval of indices of maximal cliques that contain it
- Finding the maximal cliques and ordering them!!

Interval Graphs



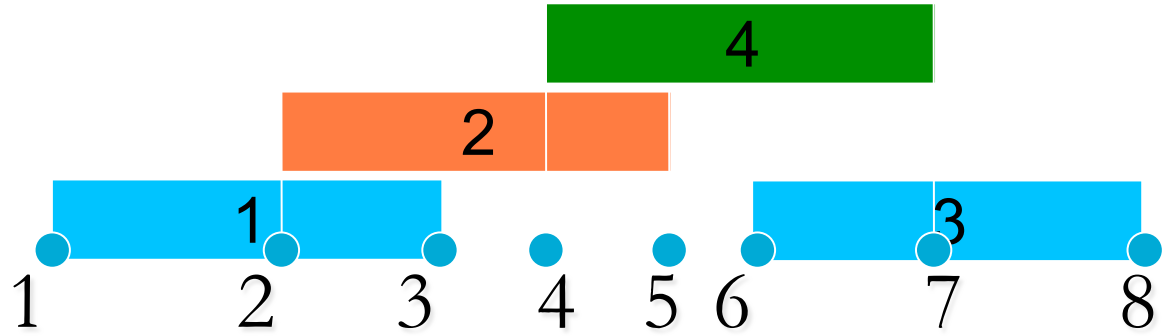
Coloring Intervals



Resouce Allocation:

- Each interval ~ Request for a resource for a period of time
- Color ~ Resource

Intervals as Paths



- Coloring intervals is same as coloring paths in linear graphs/chains.
- **Path Coloring** : Given a set of paths in a graph, assign a color to each path such that no two paths will get he same color if they have a common edge.
- Online path coloring

Types of Coloring

- Offline coloring
  - **Optimal coloring** : *Arrange the colors in some order; Assign the least possible color to each interval in non-decreasing order of their start times.*
  - chromatic number = clique number
- Online Coloring
  - First fit
  - Kierstead's algorithm

Example

Competitive Ratio

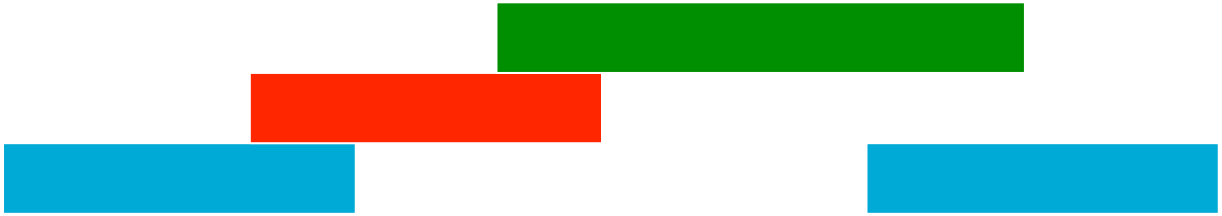
$$\text{Competitive Ratio of A} = \max_{G,s} \frac{\text{No of colors used by the online algo. A}}{\text{No of colors used by the optimal offline algorithm}}$$

$$= \max_{G,s} \frac{\text{No of colors used by the online algo. A}}{\omega}$$

First fit

- **Principle**: Consider the colors in some order and assign the least feasible color to the incoming interval.
- Simple to implement
- 8-competitive
- There exists an instance on which First fit uses at least 4.4ω colors.

First fit: Example-1



- Clique size is 2
- No of colors used is 3

First fit: Example-2

- Clique size is 2
- No of colors used is 4

Properties of First fit

- **Property**: If an interval I is colored j, then there exists an interval I' in each color i, $1 \leq i \leq j$ such that I intersects I'.
- Wall like structure

$4.4\omega \leq h \leq 8\omega$

Kierstead's Algorithm

- The best known online algorithm
- Uses at most 3ω-2 colors
- Outperformed by First fit on random instances
- Basis for designing efficient algorithms for online coloring *intervals with bandwidth*

Kierstead's Arrangement

- Each interval I is assigned a position p, such that p is the least possible position below which I is supported by a clique of size p-1.

- There can be at most ω positions.

- No interval is contained in any interval or union of intervals of the same position.

- All intervals assigned to the same position are can be colored with at most 3 colors online. Uses at most $3\omega-2$ colors.

Chordal Graphs

- A Graph in which there is no induced cycle of length four or more.
  - A 4 clique with one edge removed - chordal
  - A 4 cycle with an additional central vertex adjacent to all four - not chordal
- Every interval graph is a chordal graph
- What is the structure of chordal graph?
  - Are they intersection graphs of some meaningful collection of sets?
    - very natural question

Separators are Cliques

- In chordal graphs minimal vertex separators are cliques
  - structure of minimal separators are very important
  - Also a characterization
- Let X be a minimal u-v separator
  - Assume X is not a clique
  - Because of minimality, for each x in X, in each component (after removal of X), x has a neighbor in the component.
  - Let C1 and C2 be two components

Why? ..

- Let x1 and x2 be 2 vertices in X, not adjacent
  - Let a1 and a2 be neighbors in C1, and b1 and b2 in C2
  - Then a1 x1 b1 P' b2 x2 a2 P a1 is a cycle
  - From this cycle, we can construct a chordless cycle, contradiction
- The reverse direction
  - If all minimal separators are cliques, no induced cycles.
  - If C is an induced cycle, take x in C and y in C and take any minimal x-y separator containing the neighbors of x in C. Contradiction

Simplicial Vertices

- A vertex whose neighbor induces a clique
- An incomplete chordal graph has two non-adjacent simplicial vertices!!!
- Proof by induction in the number of vertices
  - a single vertex, is simplicial (Why?)
  - consider an edge, both are
  - consider a path, the degree 1 vertices are (base case)
  - Let X be a minimal separator
    - Consider A + X and B + X

Since X is a clique..

- apply induction to A+X and B+X
  - they are chordal and smaller.
  - A and B are non-empty
  - take nonadj v_a1, v_a2 in A+X and nonadj v_b1, v_b2 in B+X that are simplicial.
  - at most one of v_a1, v_a2 (v_b1, v_b2) can be in X
  - so we get at least 2 simplicial vertices
- What if A+X is complete, then it is easier.
  - we get a simplicial vertex from A, which is what we want.

Perfect Simplicial Ordering

- v_1, v_2, ..., v_n is a very special ordering
  - Property: higher numbered numbers of v_i induce a clique in G
- Consequence
  - Color greedily using a simplicial ordering
  - note simplicial ordering can be found in polynomial time.
- And more....

Finding the maximal cliques

- Based on a structural property of graphs that do not have induced 4 cycles.