Algorithm design in Perfect Graphs
N.S. Narayanaswamy
IIT Madras

What is it to be Perfect?

- Introduced by Claude Berge in early 1960s

- Coloring number and clique number are one and the same for all induced subgraphs of a Graph

- Note that the coloring number is at least the clique number

- Are the even unequal? - Odd cycles!!!

- To be perfect, induced subgraphs cannot be odd cycles

Exercise in Coloring

- For any given two integers, o and c, does there exist a graph whose coloring number is c and clique number is o.

- For o=2 and c=3, answer is obviously yes.

- Construct a graph for o=2 and c=4.

- Answered by Lovasz for arbitrary values of o and c.

- Check text on Graph Theory by Bondy and Murty.

Perfect Questions

- Is a given graph Perfect?

- Is there a characterization of perfect graphs?

- Is a graph minimally imperfect?

- Do any hard computational exercises become easy on these graphs?

- Are there interesting sub-classes?

This talk: A survey of the first 4 and a sample of the last question

Characterizations

- Strong Perfect Graph Theorem

A Graph is perfect if and only if it does not contain a odd cycle or its complement as an induced subgraph- last decade Chudnovsky..

- Conjectured by Berge in 1960

- A forbidden subgraph characterization.

- Conjecture settled after many years of research in the first decade of this century.

- Come up with a verification algorithm?

Results along the way

- Weak Perfect Graph Theorem [Lovasz, Fulkerson]

A Graph is perfect if and only if its complement is perfect.

Further, G is perfect if and only if for each induced subgraph H, the alpha-omega product is at least the number of vertices in H.

- Consequently, independence number is same as clique cover number for all induced subgraph of a perfect graph.

Polyhdedral Combinatorics

- Main goal-understanding the geometric structure of a solution space.

Visualize the convex hull and find a system of inequalities that specify exactly the convex hull

- Consider the convex hull of stable set incidence vectors

- Consider the clique inequalities

- G is perfect if and only if the convex hull and clique inequality polytope are identical

Summary of Survey

- Perfect graphs are motivated by coloring issues.

- Connects combinatorial understanding to polyhedral structure in a very rich and fundamental way

Geometric Algorithms and Combinatorial Optimization – Groetschel, Lovasz, Schrijver

Algorithmic Graph Theory and Perfect Graphs – Golumbic

The Sandwich Theorem – Knuth

Interval Graphs

- A subclass of perfect graphs

- Motivated by many applications

  - Temporal reasoning issues like register allocation

- Given a set of intervals, consider the natural intersection graph for which there is one vertex per interval and an edge indicates a non-empty intersection

- Examples of interval graphs and non interval graphs

Interval Graphs are perfect

- Given a graph, find an interval representation

- Visualize the intervals as time intervals

- Color the intervals in increasing order of time

- Reuse a color whenever possible and use a new color greedily

- This proves that interval graphs are perfect.

- Key issues: given a graph, does it have an interval represenation.

Forbidden subgraphs

- Induced cycles of length more than 3

- Asteroidal triples

3 vertices x, y, z form an asteroidal triple if for all ordering of them, there is a path from the first to third which avoids the neighbors of the second.

- Gives a polynomial time algorithm

  - Check no four form an induced cycle

  - Check no 3 form an asteroidal triple

The interval representation

- Graph is an interval graph if and only if its maximal cliques can be linearly ordered such that the set of maximal cliques containing a vertex occur consecutively in the order.

- Note that this consecutive ordering gives the interval representation

  - For each vertex, the interval associated is the interval of indices of maximal cliques that contain it

- Finding the maximal cliques and ordering them!!

Chordal Graphs

- A Graph in which there is no induced cycle of length four or more.
    - A 4 clique with one edge removed - chordal
    - A 4 cycle with an additional central vertex adjacent to all four - not chordal
- Every interval graph is a chordal graph
- What is the structure of chordal graph?
    - Are they intersection graphs of some meaningful collection of sets?
        - very natural question

13

Separators are Cliques

- In chordal graphs minimal vertex separators are cliques

    - structure of minimal separators are very important

    - Also a characterization

- Let X be a minimal u-v separator

    - Assume X is not a clique

    - Because of minimality, for each x in X, in each component (after removal of X), x has a neighbor in the component.

    - Let C1 and C2 be two components

14

Why? ..

- Let x1 and x2 be 2 vertices in X, not adjacent

  - Let a1 and a2 be neighbors in C1, and b1 and b2 in C2

  - Then a1 x1 b1 P' b2 x2 a2 P a1 is a cycle

  - From this cycle, we can construct a chordless cycle, contradiction

- The reverse direction

  - If all minimal separators are cliques, no induced cycles.

  - If C is an induced cycle, take x in C and y in C and take any minimal x-y separator containing the neighbors of x in C. Contradiction

15

Simplicial Vertices

- A vertex whose neighbor induces a clique

- An incomplete chordal graph has two non-adjacent simplicial vertices!!!

- Proof by induction in the number of vertices

    - a single vertex, is simplicial (Why?)

    - consider an edge, both are

    - consider a path, the degree 1 vertices are (base case)

    - Let X be a minimal separator

        - Consider A + X and B + X

Since X is a clique..

- apply induction to A+X and B+X
    - they are chordal and smaller.
    - A and B are non-empty
    - take nonadj v_a1, v_a2 in A+X and nonadj v_b1, v_b2 in B+X that are simplicial.
    - at most one of v_a1, v_a2 (v_b1, v_b2) can be in X
    - so we get at least 2 simplicial vertices
- What if A+X is complete, then it is easier.
    - we get a simplicial vertex from A, which is what we want.

17

Perfect Simplicial Ordering

- $v_1, v_2, ..., v_n$ is a very special ordering
  - Property: higher numbered numbers of v_i induce a clique in G
- Consequence
  - Color greedily using a simplicial ordering
  - note simplicial ordering can be found in polynomial time.
- And more....

18

Finding the maximal cliques

- Based on a structural property of graphs that do not have induced 4 cycles.