Duality Transformation and its Application to Computational Geometry

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Outline

Introduction

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- In this lecture we explore how geometric duality can be used to design efficient algorithms by considering a number of problems in computational geometry.
- For simplicity, we consider duality in two dimensions only. However, the concept generalizes to higher dimensions also.

Introduction

- In the Cartesian plane, a point has two parameters (x- and y-coordinates) and a (non-vertical) line also has two parameters (slope and y-intercept). We can thus map a set of points to a set of lines, and vice versa, in an one-to-one manner.

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- This natural duality between points and lines in the Cartesian plane has long been known to geometers.

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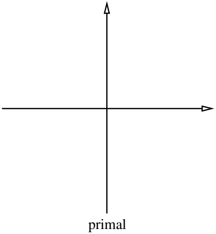
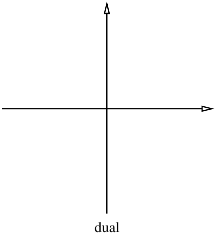
- Each such mapping has its advantages and disadvantages in particular contexts.
- In this lecture we define a particular form of duality and explore its properties and applications.

Outline

Definition

Let D be the duality transformation.





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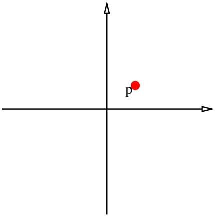
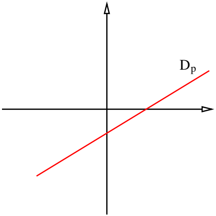
A point $p(a, b)$ is transformed to the line $D_p(y = ax - b)$.



primal



dual

Definition

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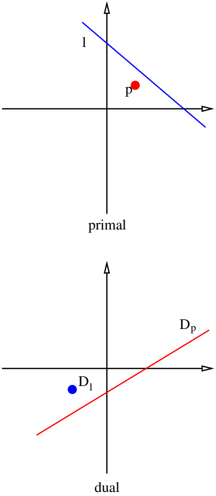
Definition

A point $p(a, b)$ is transformed to the line $D_p(y = ax - b)$.

Definition

A line $l(y = cx + d)$ is transformed to the point $D_l(c, -d)$.



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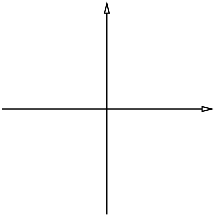
However this is not a problem in general. Because we can always rotate the problem space slightly so that no line is vertical. Sometimes, vertical lines are taken as special cases and treated separately.

Properties

Incidence is preserved



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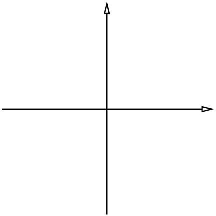


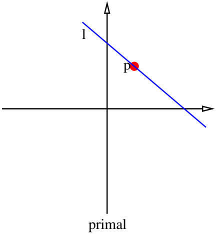
dual

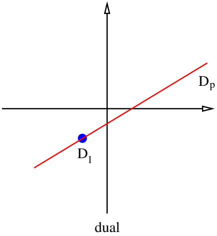
Properties

Incidence is preserved

Lemma

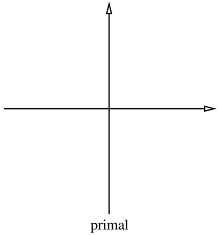
A point $p(a, b)$ is incident to the line $l(y = cx + d)$ in the primal plane iff point $D_l(c, -d)$ is incident to the line $D_p(y = ax - b)$ in the dual plane.

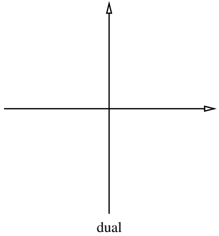




Properties

But order is reversed





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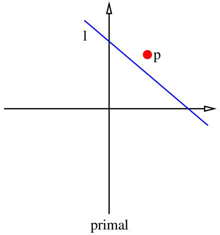
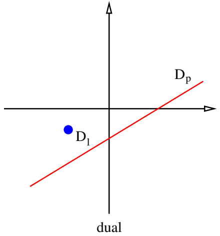


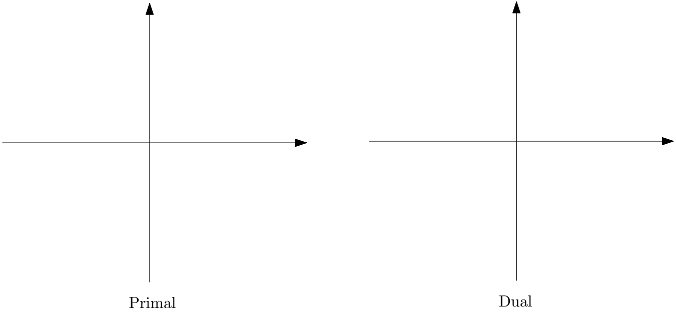
Lemma

A point $p(a, b)$ is above (below) the line $l(y = cx + d)$ in the primal plane iff line $D_p(y = ax - b)$ is below (above) the point $D_l(c, -d)$ in the dual plane.



Example1

- Given a point in the primal plane, find its dual.



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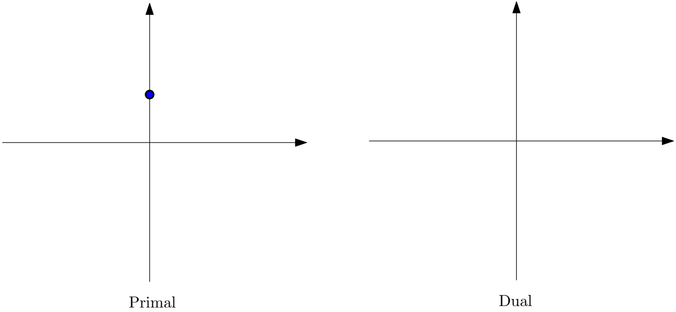

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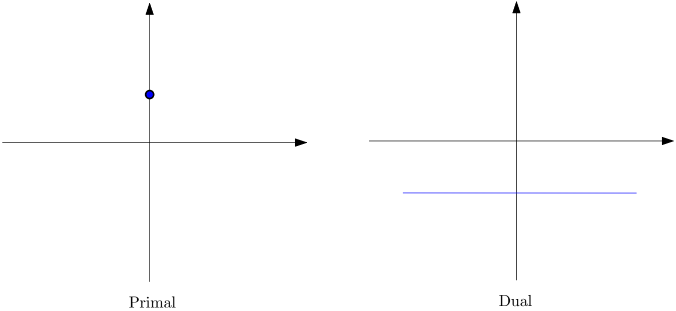

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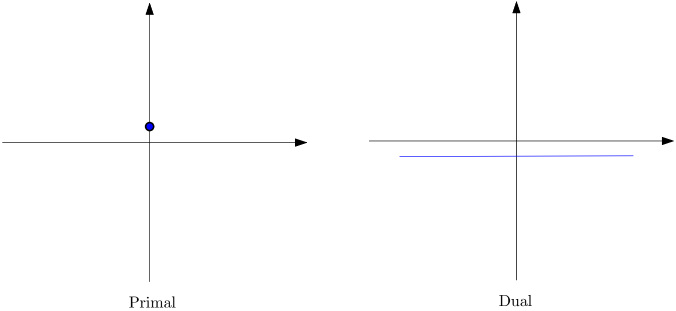

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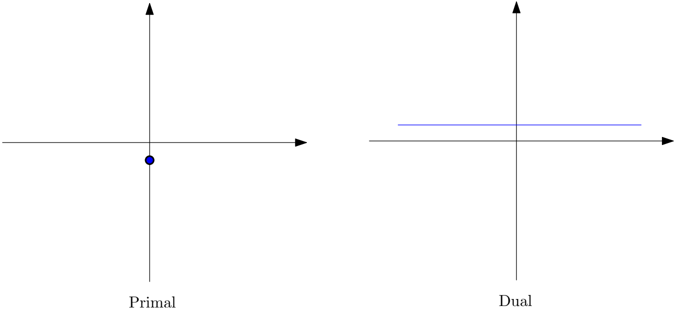
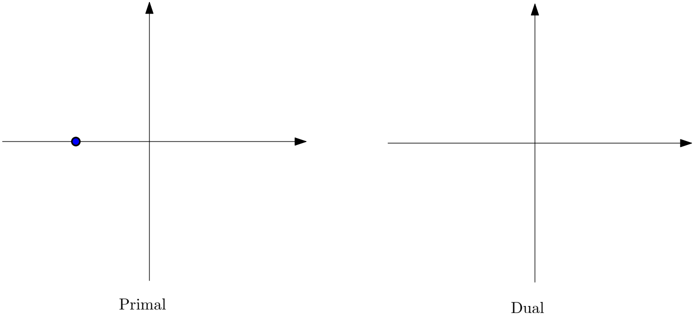


Example2

- Given a point in the primal plane, find its dual.



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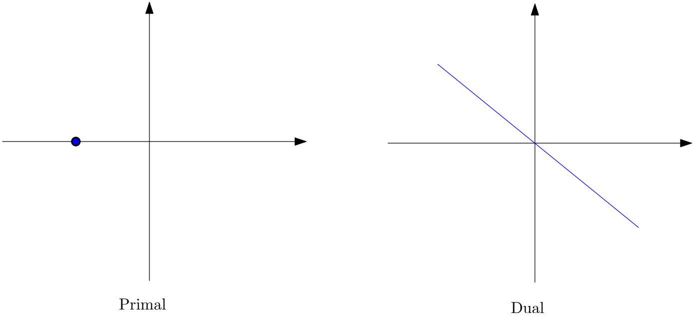

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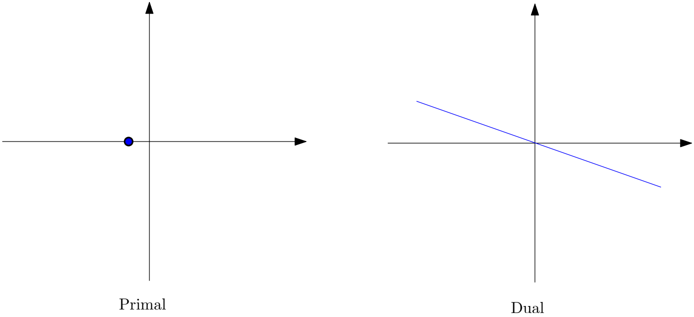

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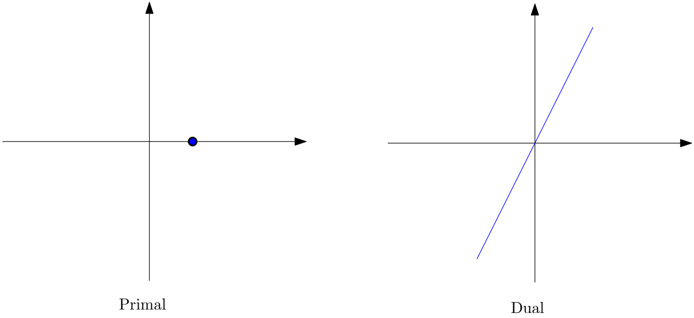
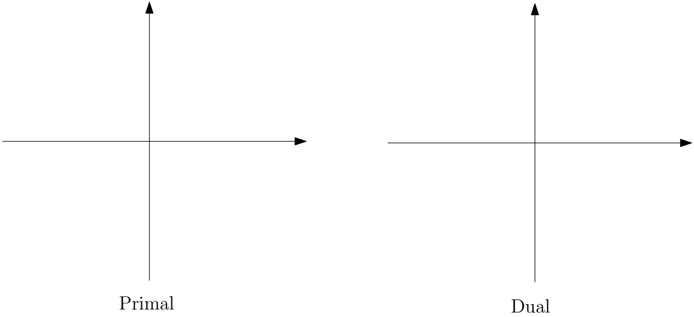


Example3



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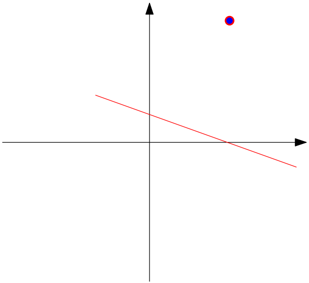
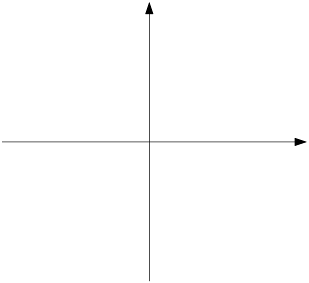
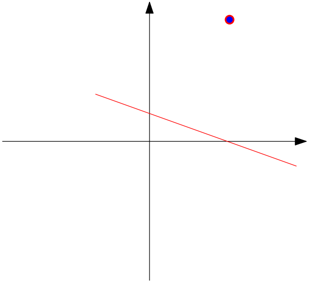


Primal



Dual

Example3



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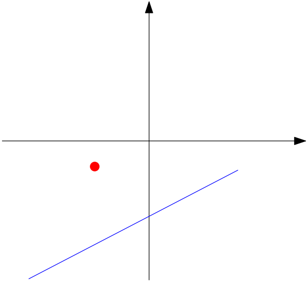

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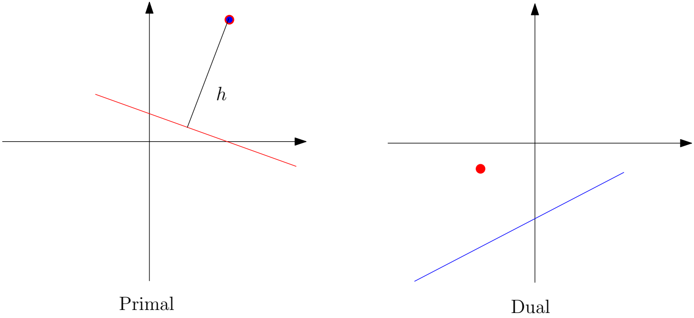

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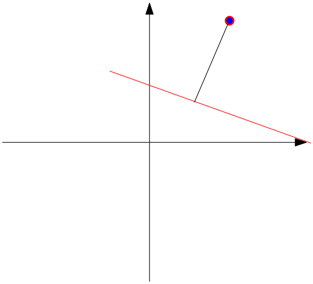

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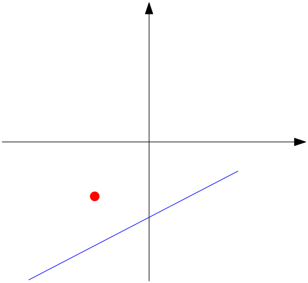

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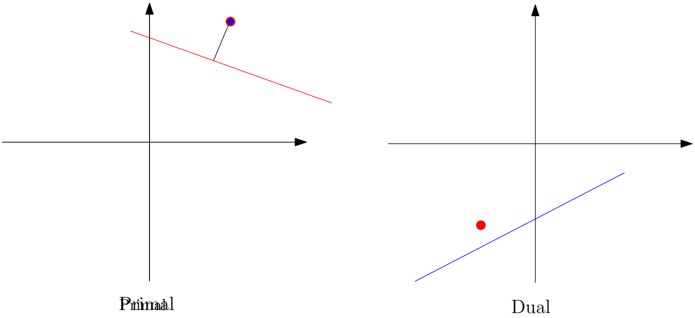

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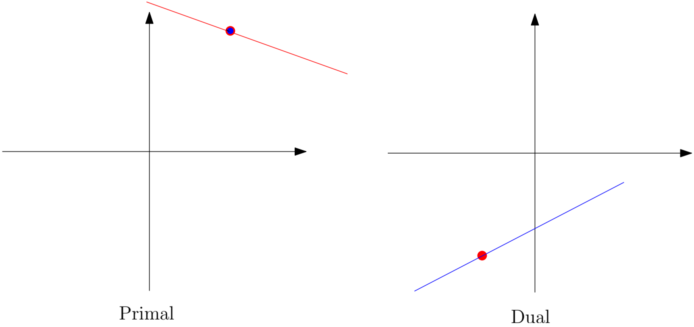

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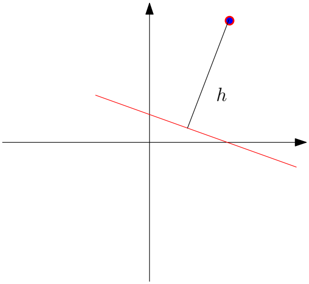

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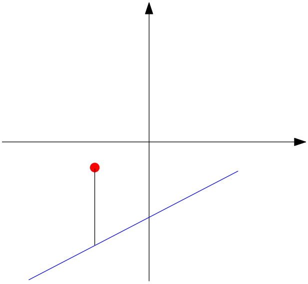

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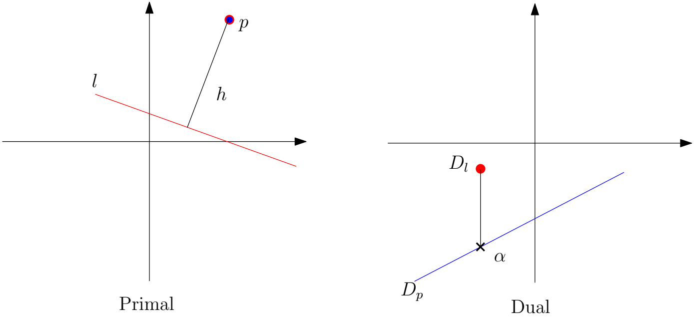

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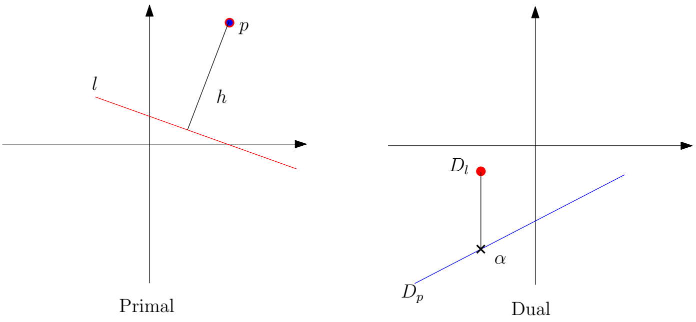

$$h = \frac{d(D_l, \alpha)}{\sqrt{1 + (x(D_l))^2}}$$

Here $d(.,.)$ is distance between two points.

And $x(.)$ is x-coordinate of a point.

Alternative Definition

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- There are variations of m-c duality. For example, a variation of m-c duality is: $p(a, b) \rightarrow D_p(y = ax + b)$ and $l(y = cx + d) \rightarrow D_l(-c, d)$. Observe that, here $DD_p \neq p$ and $DD_l \neq l$, but both incidence and order are preserved.

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- An alternative definition, called polar duality, is also used.

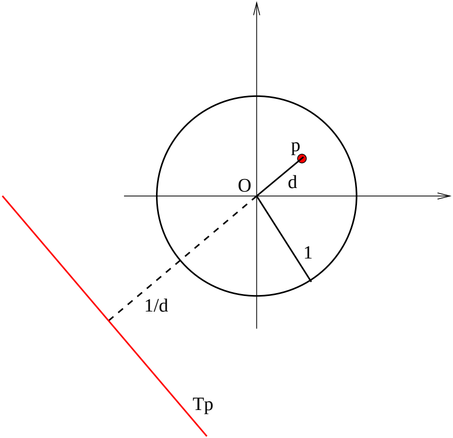
Polar Duality

Definition

A point p with coordinates (a, b) in the primal plane corresponds to a line T_p with equation $ax + by + 1 = 0$ in the dual plane and vice versa.

Polar Duality

- Geometrically this means that if d is the distance from the origin(O) to the point p, the dual T_p of p is the line perpendicular to Op at distance $1/d$ from O and placed on the other side of O.



Outline

Convex Hull Problem

Problem

Given a set $\mathcal{P}$ of points in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$.

Optimal Algorithms

- By reducing the sorting problem to the convex hull problem, it can be shown that the worst case computational complexity of the convex hull problem is $O(n \log n)$, where n is the size of the given point set.

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- A number of optimal algorithms have been devised for the convex hull problem.

Optimal Algorithms

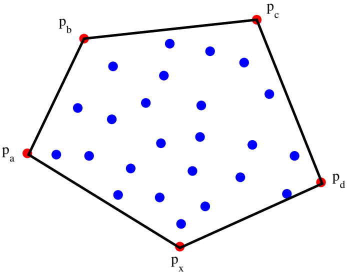
- Grahams scan, time complexity $O(nlogn)$.
  (Graham, R.L., 1972)
- Divide and conquer algorithm, time complexity $O(nlogn)$.
  (Preparata, F. P. and Hong, S. J., 1977)
- Jarvis's march or gift wrapping algorithm, time complexity $O(nh)$ where h number of vertices of the convex hull.
  (Jarvis, R. A., 1973)
- Most efficient algorithm to date is based on the idea of Jarvis's march, time complexity $O(nlogh)$.
  T. M. Chan (1996)

An Optimal Algorithm using Duality

- We now develop an optimal algorithm for computing convex hull using the concept of duality.

Definitions

Let $\mathcal{P}$ be the given set of n points in the plane. Let $p_a \in \mathcal{P}$ be the point having smallest x-coordinate and $p_d \in \mathcal{P}$ be the point with largest x-coordinate. Obviously, both p_a and p_d belongs to $CH(\mathcal{P})$.



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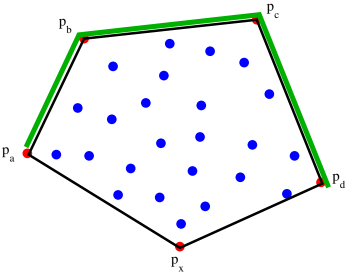
Definition

The c-wise polygonal chain $p_a, \ldots, p_d$ along the hull is called the upper hull.



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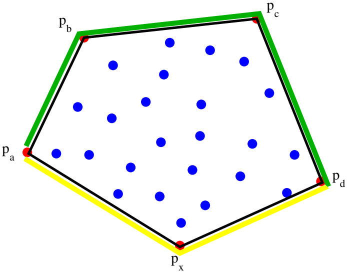
Definition

The c-wise polygonal chain $p_a, \ldots, p_d$ along the hull is called the upper hull.

Definition

The cc-wise polygonal chain $p_a, \ldots, p_d$ along the hull is called the lower hull.



Definitions

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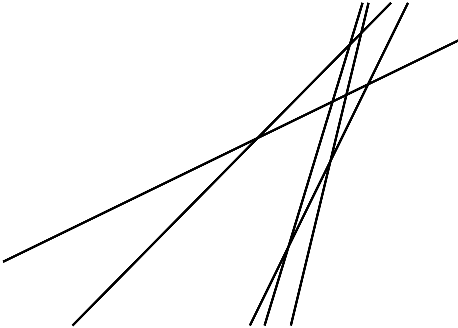

Definitions

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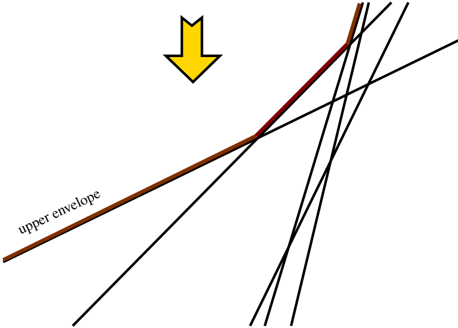
Definition

The upper envelope is a polygonal chain E_u such that no line $l \in \mathcal{L}$ is above E_u.



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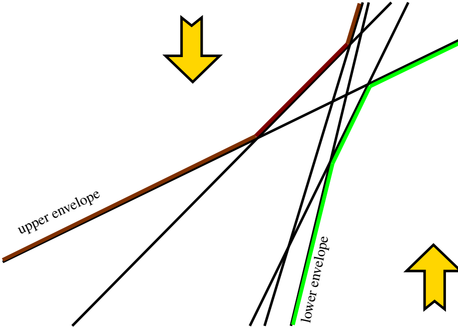
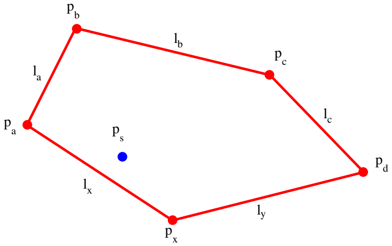
Definition

The lower envelope is a polygonal chain E_l such that no line $l \in \mathcal{L}$ is below E_l.



Connection Between Hull and Envelope



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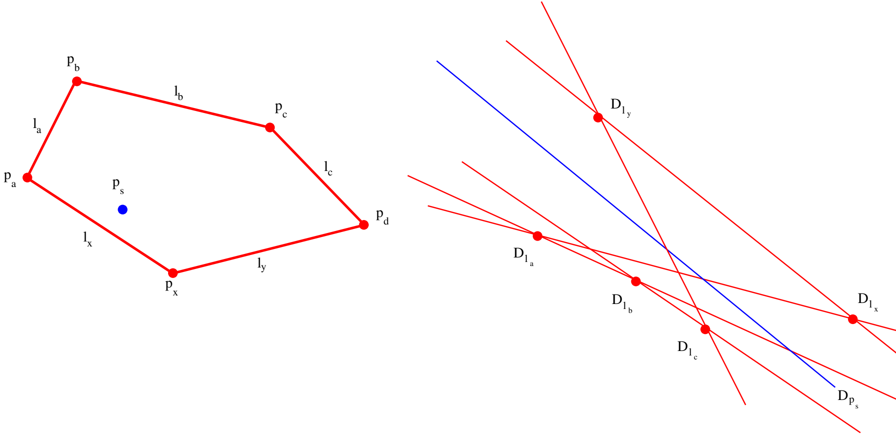

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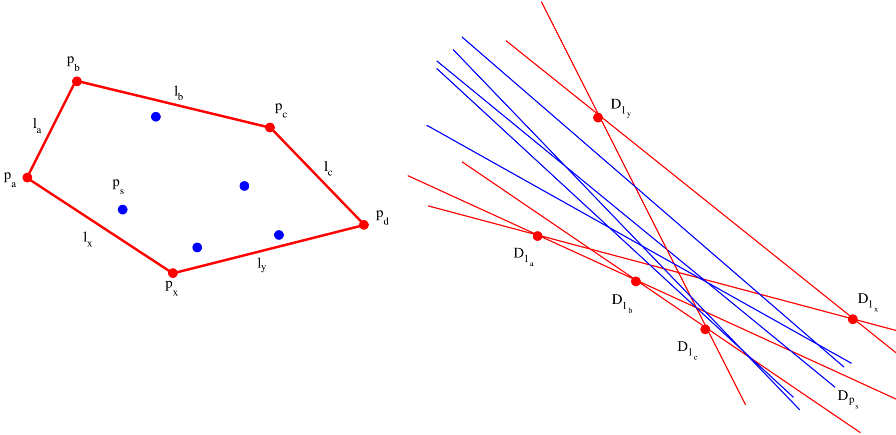

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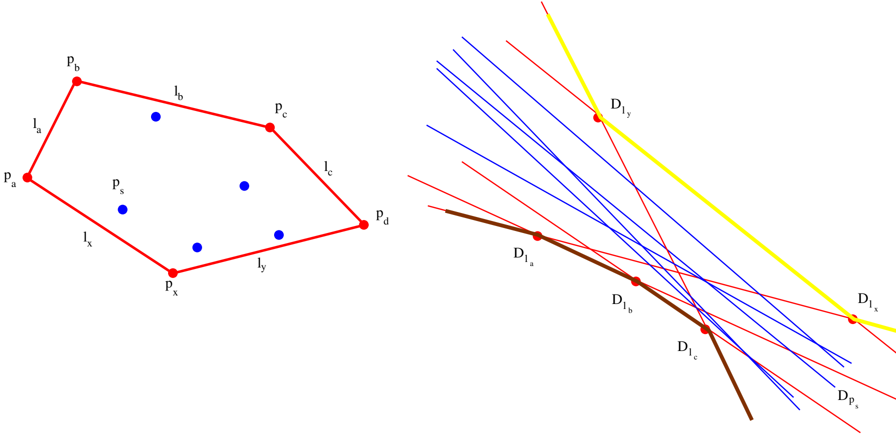

Connection Between Hull and Envelope

Conclusion

Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.

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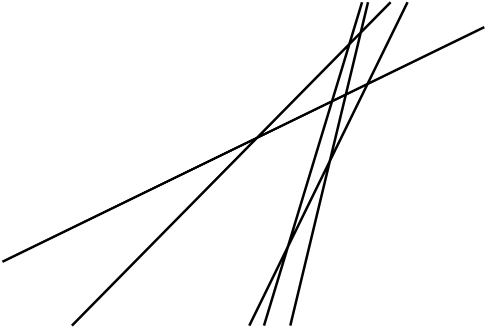
Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.

Thus the problem of computing convex hull of a point set in the primal plane reduces to the problem of computing upper and lower envelopes of the corresponding set of lines in the dual plane.

Outline of the algorithm



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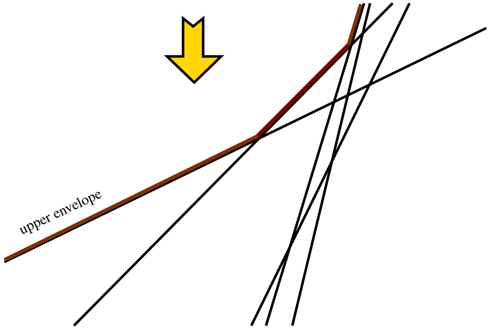

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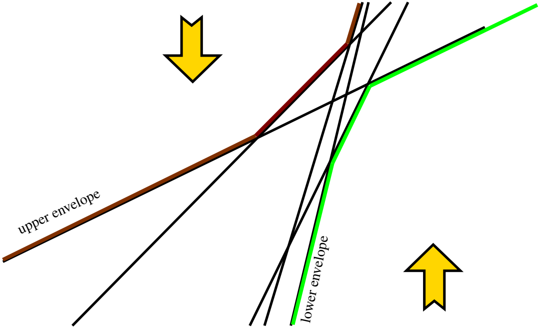
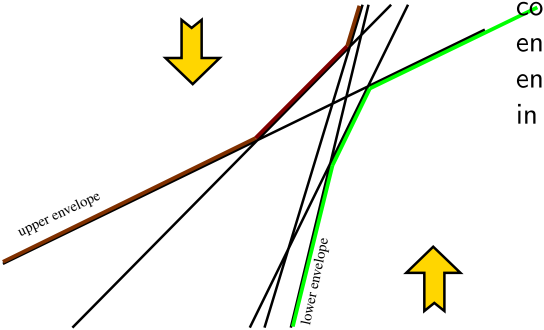


Outline of the algorithm



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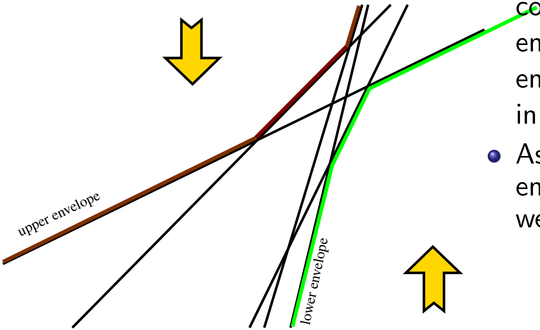

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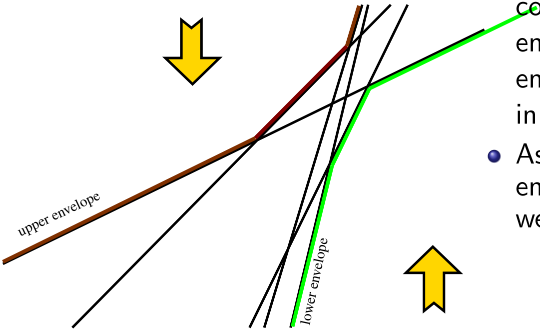

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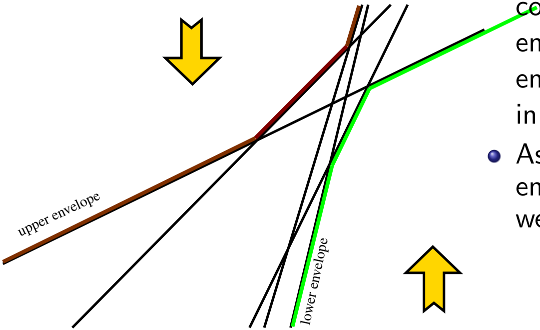

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- As we scan the upper envelope from left to right, we notice that:
    - The line with smallest slope is always present as the first member.
    - Slopes of the members are in increasing order.

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Input: I = (L1, L2, ..., Ln) is the list of lines
       in the increasing order of slopes.
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  insert the line segment Li at the tail of the list O;
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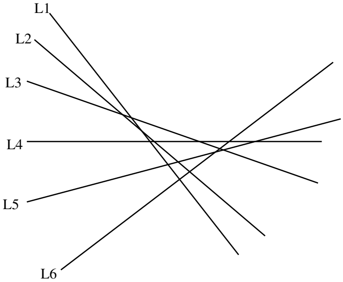
```

Example



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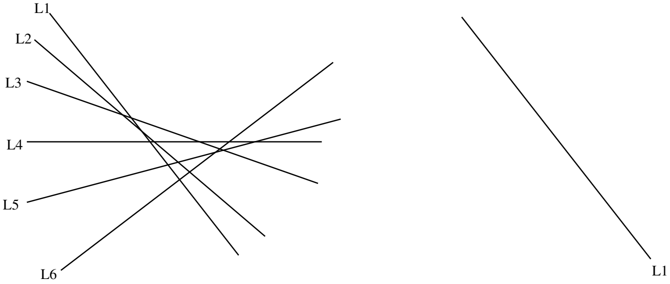

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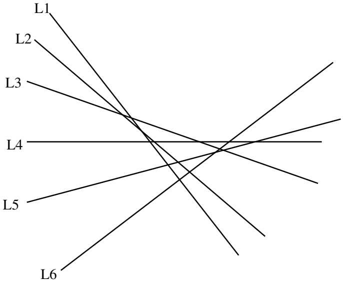

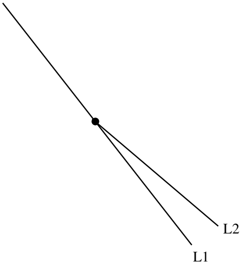

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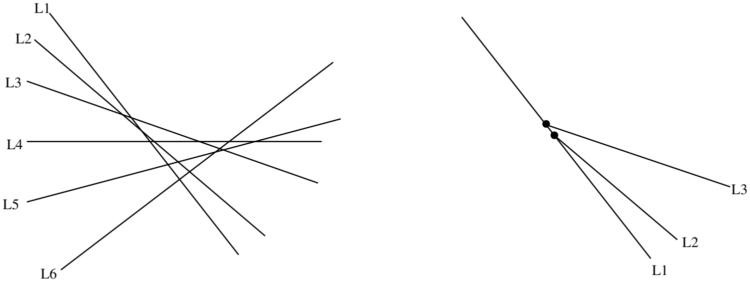

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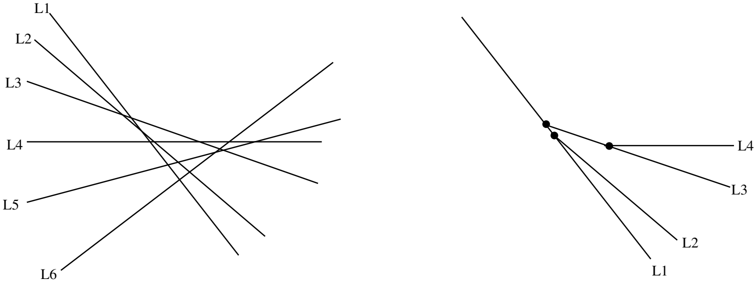

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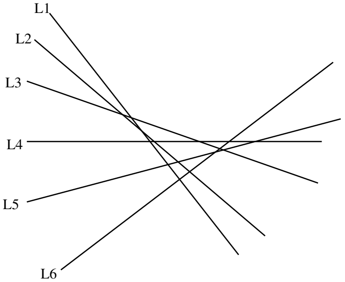

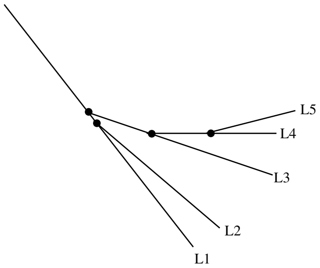

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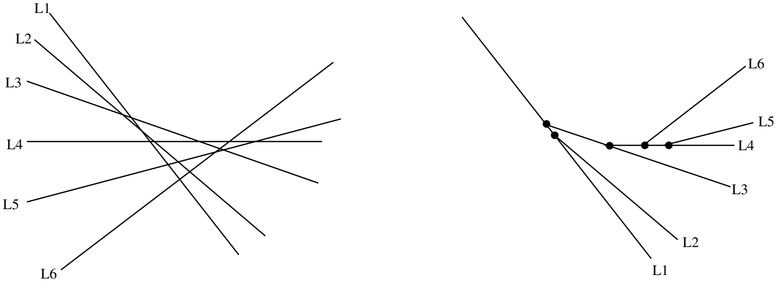

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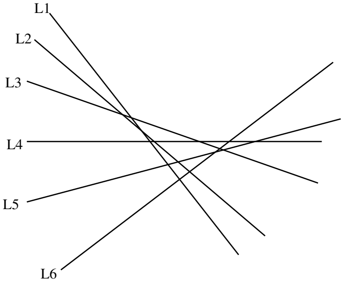

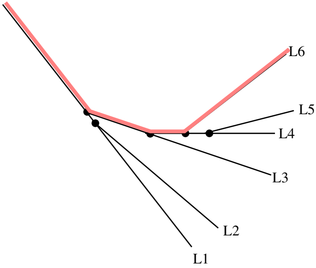

Result

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After sorting n lines by their slopes in $O(nlogn)$ time, the upper envelope can be obtained in $O(n)$ time.

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After sorting n lines by their slopes in $O(nlogn)$ *time, the upper envelope can be obtained in* $O(n)$ *time.*

Proof.

It may check more than one line segment when inserting a new line, but those ones checked are all removed except the last one. ☐

Result

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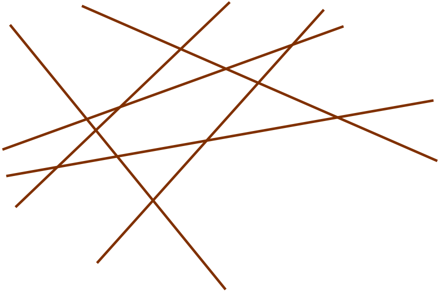
Given a set $\mathcal{P}$ of n points in the plane, $CH(\mathcal{P})$ can be computed in $O(n \log n)$ time using $O(n)$ space.

Outline

Definition

Definition

Let $\mathcal{L}$ be a set of n lines in the plane. The embedding of $\mathcal{L}$ in the plane induces a planner subdivision that consists of vertices, edges, and faces where some of the edges and faces are unbounded. This subdivision is referred to as arrangement induced by $\mathcal{L}$, and is denoted by $A(\mathcal{L})$.



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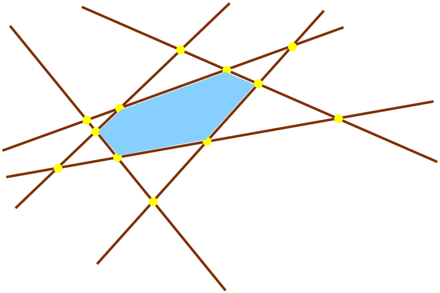

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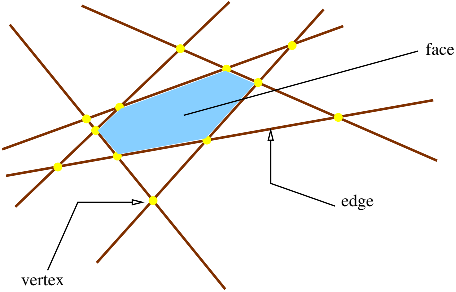

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Observation

Worst case complexity occurs when an arrangement is simple.

Result

Theorem

Let $\mathcal{L}$ be the set of n lines in the plane, and let $A(\mathcal{L})$ be the arrangement induced by $\mathcal{L}$.

(i) *The number of vertices of $A(\mathcal{L})$ is at most $n(n-1)/2$.*

(ii) *The number of edges of $A(\mathcal{L})$ is at most n^2.*

(iii) *The number of faces of $A(\mathcal{L})$ is at most $n^2/2 + n/2 + 1$.*

Equality holds in these three statements iff $A(\mathcal{L})$ is simple.

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Can be proved easily by using Euler's formula:
For any connected planner embedded graph with m_v vertices, m_e edges, and m_f faces the following relation holds

$$m_v - m_e + m_f = 2.$$

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- Algorithms for a large number of problems are based on constructing and analyzing the arrangement of a specific set of lines.
- A variety of data structures and algorithm have been proposed for this purpose.

Result

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Given a set $\mathcal{L}$ of n lines in the plane, the arrangement $A(\mathcal{L})$ induced by $\mathcal{L}$ can be constructed in $O(n^2)$ time.

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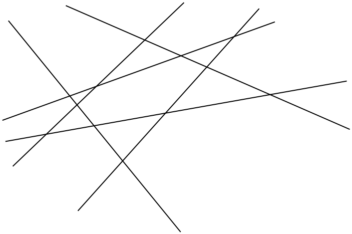
Let $\mathcal{L}$ be a set on n lines in the plane inducing an arrangement $A(\mathcal{L})$. A point π in the plane is at level θ $(0 \leq \theta < n)$ if there are exactly θ lines in $\mathcal{L}$ that lie strictly below π. The θth-level of $A(\mathcal{L})$ is the closure of a set of points on the lines of $\mathcal{L}$ whose levels are exactly θ in $A(\mathcal{L})$. We denote θth-level by λ_θ.



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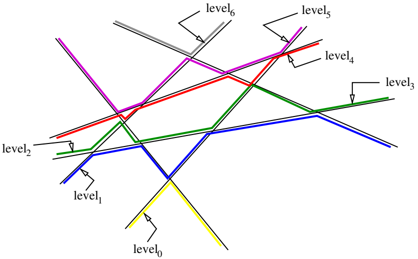

Observations

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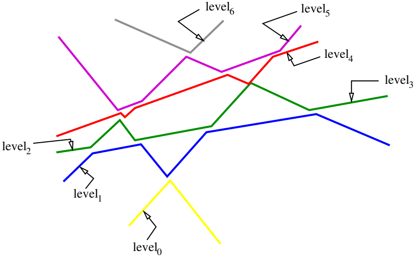

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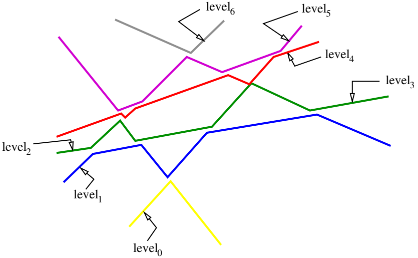

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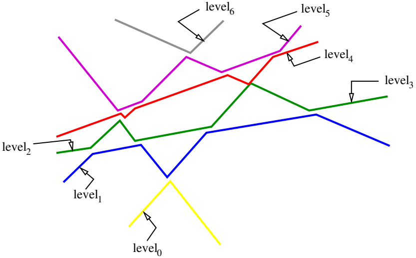

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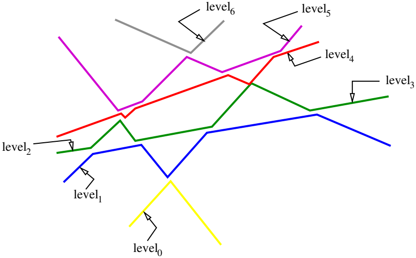
- We can thus store each level simply as an array of segments.



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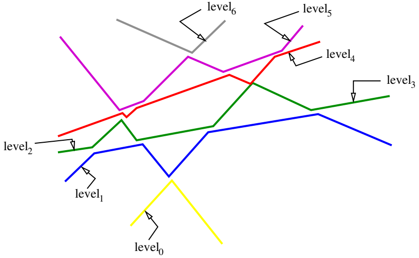

- We can thus store each level simply as an array of segments.
  - Observe that the upper and the lower envelops mentioned earlier, are simply the 0-th and $(n-1)$-th levels respectively.

Computing Levels

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- Here we consider an alternative method using plane sweep paradigm.
- The method was first introduced by Bentley and Ottmann (1979) in the context of solving the problem of line segment intersections.

Plane Sweep Method

Basic method consists of:

- A vertical line l, called the sweep line, sweeps over the arrangement from $x = -\infty$ to $x = \infty$. Observe that, at every instant, the sweep line intersects each element of $\mathcal{L}$.

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- The status changes only when the sweep line crosses vertices of the arrangement which are intersection points of pairs of lines. These intersection points are called event points.
- The algorithm performs some computational steps when the sweep line reaches event points. Specifically, it updates the sweep line status and discover more event points to be processed subsequently.

Data structure

- Apart from storing the events, we also need to insert new events and extract the event nearest to the sweep line on its right. Clearly, a heap is a suitable data structure for this.

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- We order the lines from bottom to top according to their intersections with the sweep line. Data structure we use for maintaining the sweep line status are arrays storing the levels. At an instant, portion of the line at the i-th position, $0 \leq i < n$, is part of the i-th level.

Processing

- Let the next event be the intersection point of the lines currently at i-th and $(i+1)$-th positions respectively. Processing steps to be performed at this event point are as follows.

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- If the line at the $(i+1)$-th position after the event point intersect the line at the $(i+2)$-th position on the right of the sweep line, then we insert the intersection point in the heap as a future event point. Similarly, if the line at the i-th position after the event point intersect the line at the $(i-1)$-th position on the right of the sweep line, then we insert this intersection point also as a future event point.

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- We first compute the left most intersection point and position the sweep line before this intersection point. This step takes $O(n^2)$ time.

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- We then initialize the level arrays with the lines according to their position on the sweep line. This step needs $O(n)$ time.
- Finally, we check each pair of lines from bottom to top if they insert on the right of the sweep line. If yes, insert these intersection points in the heap as an event point. This step needs $O(n)$ time.

Algorithm

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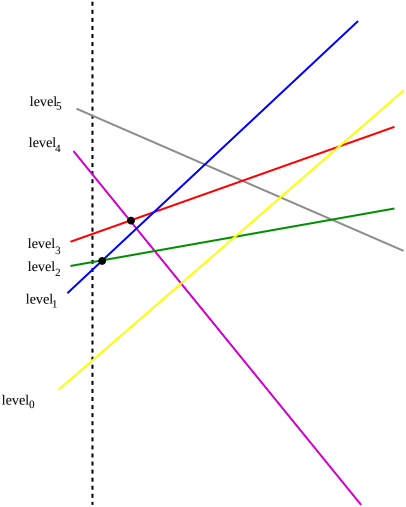
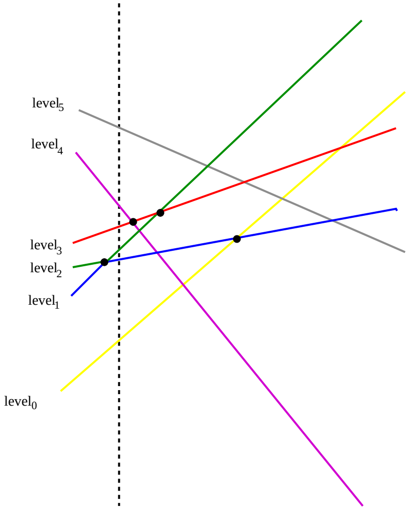
Example



Example



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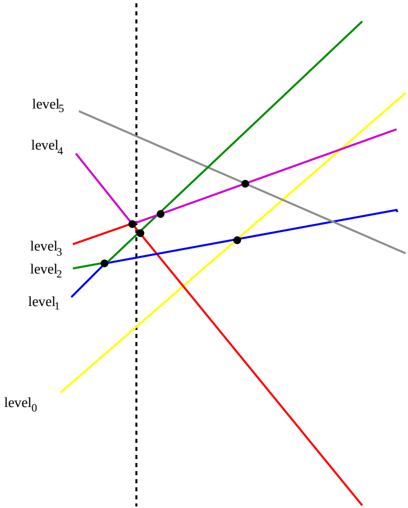

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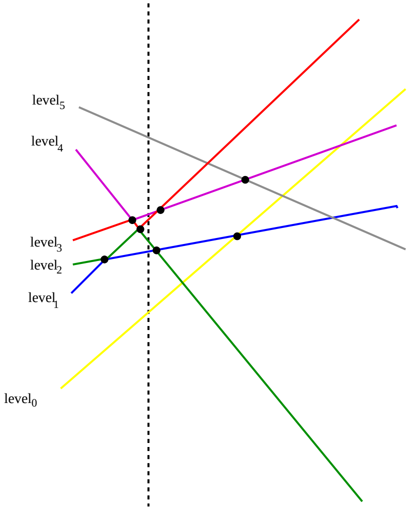
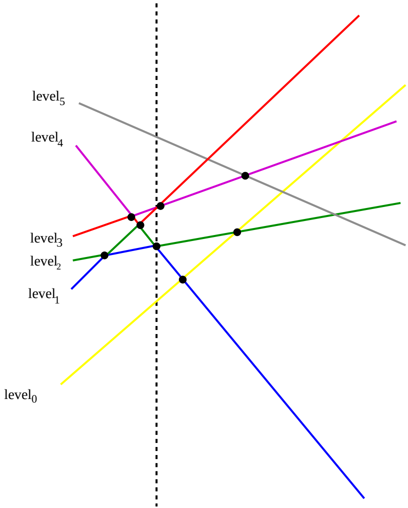
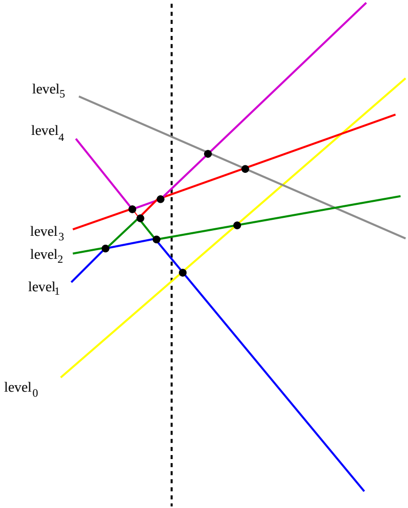

level_5
level_4
level_3
level_2
level_1
level_0

Example



Example



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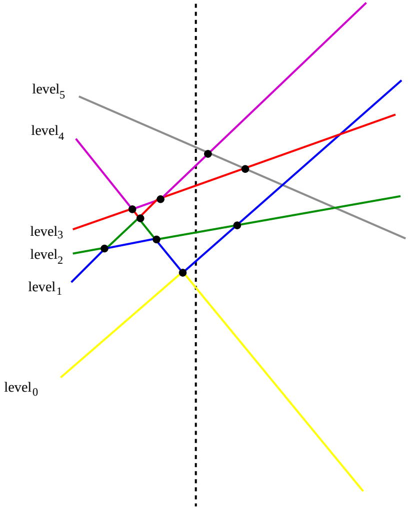
level_5
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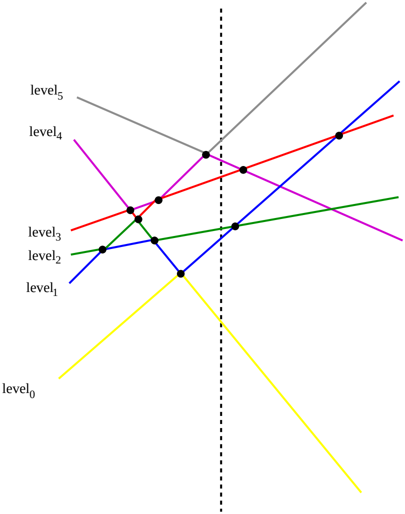
Example



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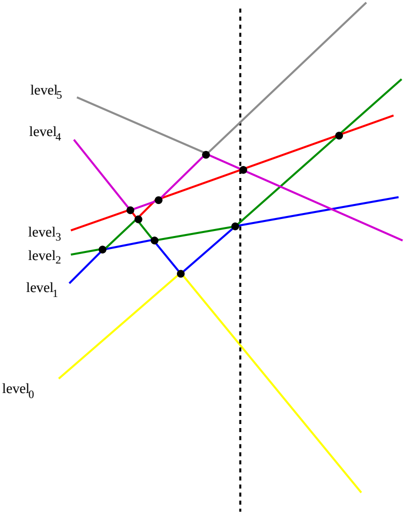
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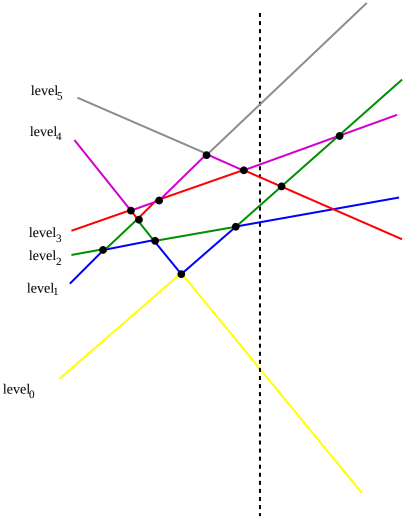
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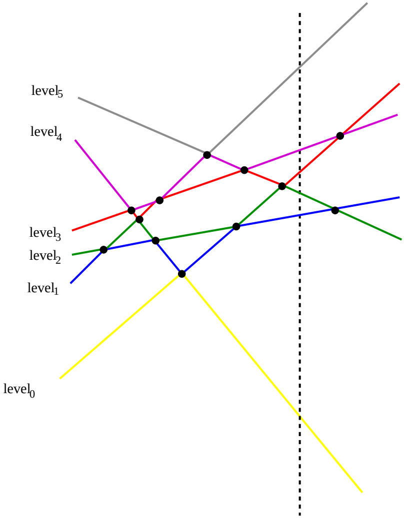

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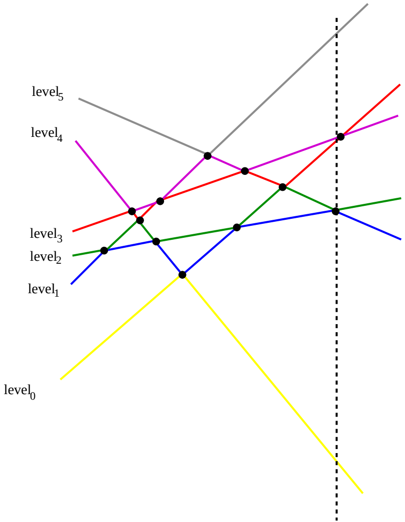
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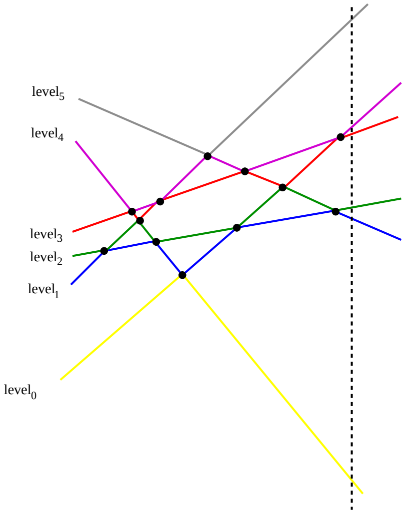
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- Space complexity is $O(n^2)$.

Result

Theorem

Using plane sweep, levels of an arrangement of n lines can be computed in $O(n^2 \log n)$ *time using* $O(n^2)$ *space.*

Outline

1. Introduction
2. Definition and Properties
3. Convex Hull
4. Arrangement of Lines
5. **Smallest Area Triangle**
6. Nearest Neighbor of a Line

Smallest Area Triangle Problem

Problem

Let $\mathcal{P}$ be a set of n points in the plane. Determine which of the $\binom{n}{3}$ triangles with vertices in $\mathcal{P}$ has the smallest area.

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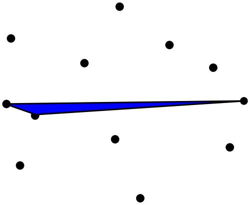
Problem

Let $\mathcal{P}$ be a set of n points in the plane. Determine whether three points in $\mathcal{P}$ are collinear.

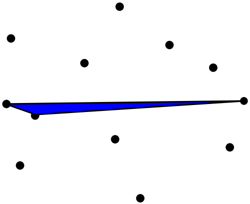
Smallest Area Triangle Problem

- The difficulty of the problem arises from the fact that the vertices of the smallest triangle can be arbitrarily apart (i.e., absence of locality).



Smallest Area Triangle Problem

- The difficulty of the problem arises from the fact that the vertices of the smallest triangle can be arbitrarily apart (i.e., absence of locality).
- A simple solution is to compute area of all possible triangles and report the one having minimum area. This scheme requires $O(n^3)$ time.



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- The best known algorithm, without using duality, for this problem has time and space complexities $O(n^2 \log n)$ and $O(n)$ respectively.
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- The best known algorithm, without using duality, for this problem has time and space complexities $O(n^2 \log n)$ and $O(n)$ respectively.
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- Using duality, it is possible to improve upon the complexity.

Assumption

- The definition of duality implies that if two points p_i and p_j in the primal plane have same x-coordinate values, then corresponding duals D_{p_i} and D_{p_j} are parallel in the dual plane.

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- To avoid this we assume that no two points in $\mathcal{P}$ have same x-coordinates. This may possibly require rotating the axes by a small angle which can be determined in $O(n \log n)$ time.

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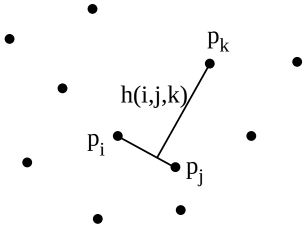

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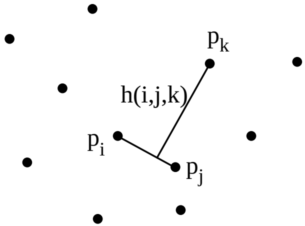
- How do we use duality to solve this problem?
- Let $h(i, j, k)$ be the perpendicular distance from the point p_k to the segment $p_i p_j$.
- Smallest area triangle with $p_i p_j$ as an edge minimizes $h(i, j, k)$ for all $k \neq i, j$; $1 \leq k \leq n$.



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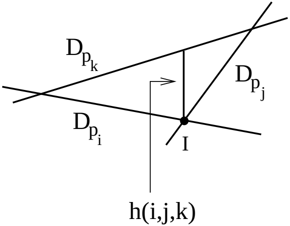
- Straight forward use of this scheme leads to an $O(n^3)$ time algorithm.
- However, when taken to dual plane, this scheme leads to an efficient algorithm.

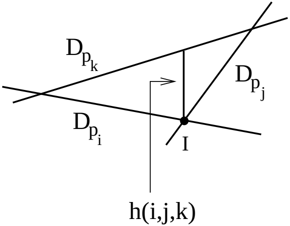
Dualization

- In the dual plane, the edge $p_i p_j$ becomes the intersection point I of D_{p_i} and D_{p_j}.



Dualization

- In the dual plane, the edge $p_i p_j$ becomes the intersection point I of D_{p_i} and D_{p_j}.
- The perpendicular from p_k on the edge $p_i p_j$ becomes vertical line segment from I to D_{p_k}.



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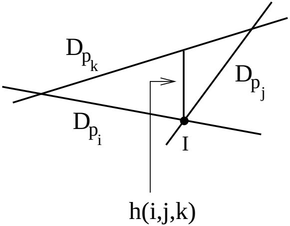
- Knowing this vertical distance in the dual plane, the perpendicular distance in the primal plane can be computed.



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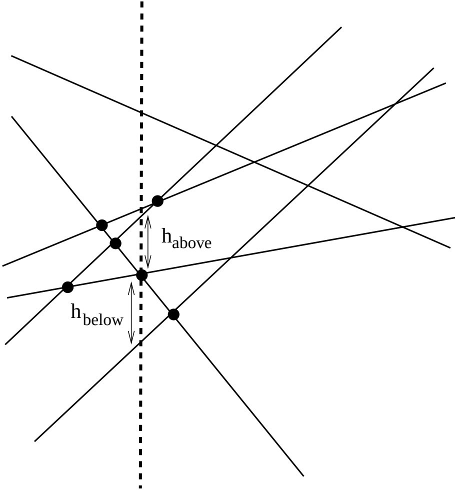
- Here event points are the intersection points between pairs of lines.

Algorithm

- When sweep line reaches an event point, the intersection point between D_{p_i} and D_{p_j} say, compute the vertical distances, along the sweep line, between the event point and the lines just above and below it.



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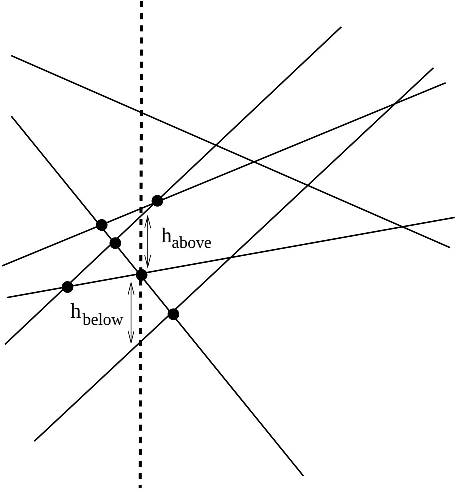
- Let the minimum distance occurs for the line D_{p_k}. Compute the minimum area of the triangle with $p_i p_j$ as base.



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- Hence space complexity of the algorithm is $O(n)$.
- Time complexity of the algorithm is, clearly, $O(n^2 \log n)$.
- The $\log n$ factor in the time complexity can be avoided by using another form of sweep line paradigm, called topological line sweep.
  (Edelsbrunner, H. and Guibas, L. J., 1989)

Outline

Problem

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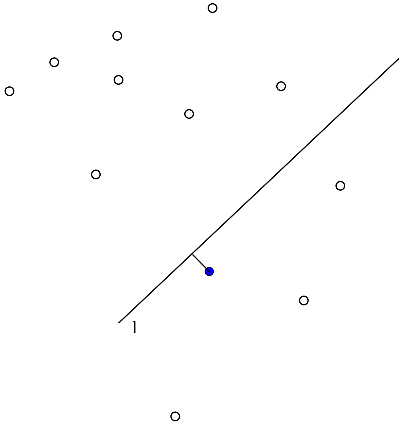
Given a set $\mathcal{P}$ of n points in the plane and a query line l, compute the nearest neighbor (in the perpendicular distance sense) of the query line l.

Problem



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- Here we are allowed to preprocess the point set so that each query can be answered efficiently.

Strategy

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- Since our definition of duality does not allow vertical line, we need to have separate algorithm for handling vertical query lines.

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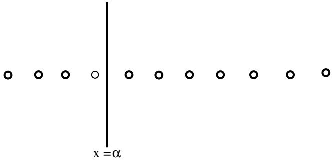

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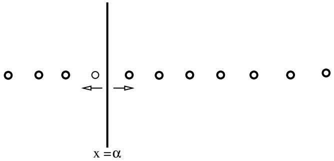

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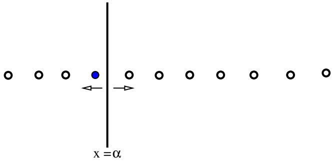
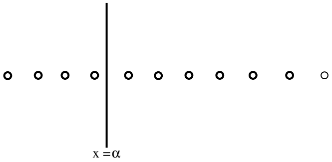


Farthest Neighbor Vertical Query Line

- Same scheme can also be used for determining the farthest neighbor of a query vertical line.



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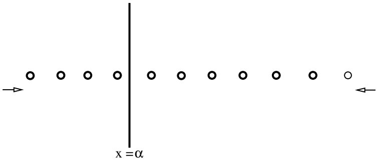

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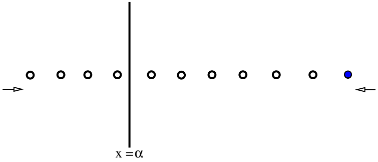

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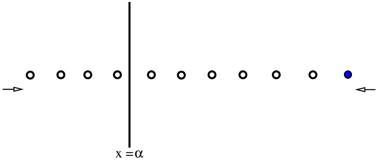

Result

Lemma

With $O(n \log n)$ *preprocessing time using* $O(n)$ *space, nearest and farthest neighbors of a query vertical line can be found in* $O(\log n)$ *time.*

Farthest Neighbor of a Non-Vertical Query Line

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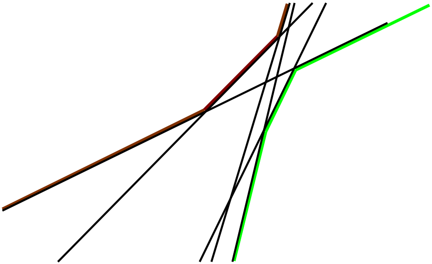
Farthest Neighbor of a Non-Vertical Query Line

- Suppose the problem is to report the farthest neighbor of a given query line which is non-vertical.
- As the preprocessing step, compute the upper envelope and the lower envelope of the set of lines dual to the given set of points $\mathcal{P}$. This can be done in in $O(n \log n)$ time using $O(n)$ space as mentioned previously.

Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.



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- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.
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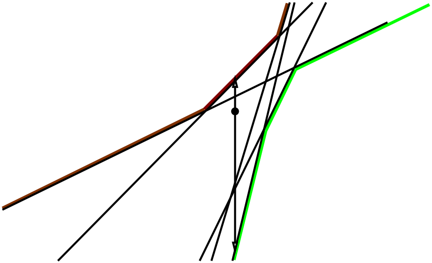
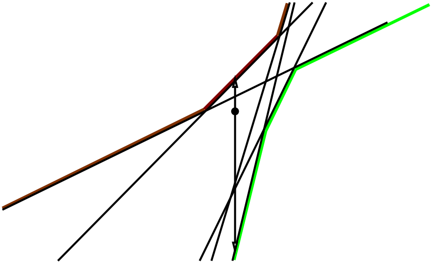


Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.
- Given a query line l, shoot a vertical ray from the point D_l in upward and downward direction and find the intersection points with the upper and the lower envelope respectively.
- This can be done in $O(\log n)$ time by using two binary searches on the arrays E_u and E_l holding the envelopes.



Result

Lemma

With $O(n \log n)$ *preprocessing time using* $O(n)$ *space, farthest neighbors of a query non-vertical line can be found in* $O(\log n)$ *time.*

Nearest Neighbor of a Query Non-vertical Line

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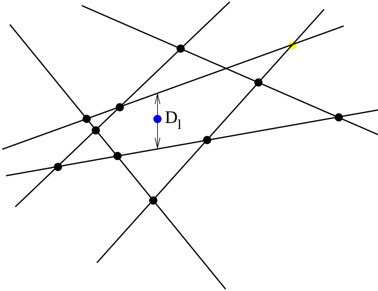
- Let $A(\mathcal{L})$ be the arrangement of lines of the set $\mathcal{L}$.
- Let f be the cell of the arrangement $A(\mathcal{L})$ containing D_l.
- Then one of the points corresponding to the lines just above D_l is the nearest neighbor of l in the primal plane.



Point Location Problem

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- So with standard data structure, nearest neighbors of a non-vertical query line can be determined in $O(\log n)$ time. The required preprocessing time and space is $O(n^2)$.
- Here we describe an algorithm for point location using levels of arrangement.

Point Location Using Level Structure

- First compute the levels of the arrangement $A(\mathcal{L})$ in $O(n^2 \log n)$ time using $O(n^2)$ space.

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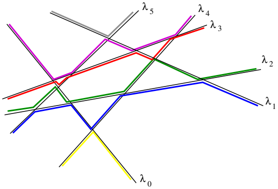
- Let λ_θ be the linear array containing vertices and edges of level θ, $\theta = 0, 1, \ldots, (n-1)$, of the arrangement $A(\mathcal{L})$.

Point Location Using Level Structure

- Create a balanced binary search tree T, called the primary structure, whose nodes correspond to the levels θ, $0 \leq \theta < n$. Each node of T, representing a level θ, is attached with the corresponding array λ_θ, called the secondary structure. This construction requires $O(n \log n)$ time and $O(n)$ space.



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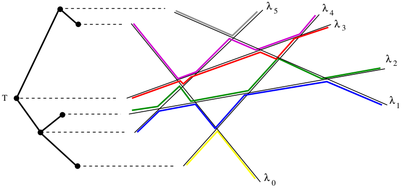
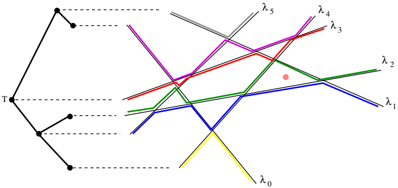


Point Location Using Level Structure

- Given the query line l, we perform two level binary search on the tree T with the point D_l.



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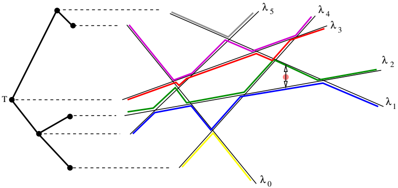
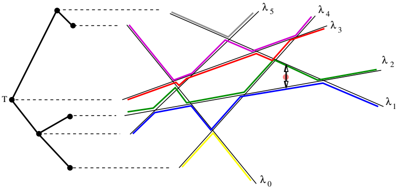
- Given the query line l, we perform two level binary search on the tree T with the point D_l.
- This will enable us to locate the two edges just above and below D_l.



Point Location Using Level Structure

- Given the query line l, we perform two level binary search on the tree T with the point D_l.
- This will enable us to locate the two edges just above and below D_l.
- Time complexity for performing this point location is $O(\log^2 n)$.



Complexity

Lemma

With $O(n^2 \log n)$ *preprocessing time and* $O(n^2)$ *space, nearest neighbor of a non-vertical query line can be determined in* $O(\log^2 n)$ *time.*

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Lemma

With $O(n^2 \log n)$ preprocessing time and $O(n^2)$ space, nearest neighbor of a non-vertical query line can be determined in $O(\log^2 n)$ time.

- It may be mentioned that the query time complexity can be reduced to $O(\log n)$, by using a data structuring technique, called fractional cascading.
  (Lueker, G. S., 1978)

Thank you!