Graph Colorings

Niranjan Balachandran

Department of Mathematics
Indian Institute of Technology Bombay.

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What is Graph Coloring?

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Chromatic number of G: *The minimum* k *such that there is a* k*-coloring of* G.

The Chromatic number is denoted by $\chi(G)$.

Example: The Petersen Graph

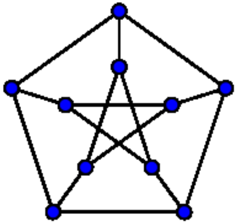

Figure : The Petersen Graph

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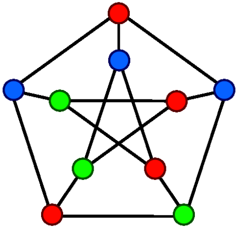

Figure : Petersen Graph with a 3-coloring.

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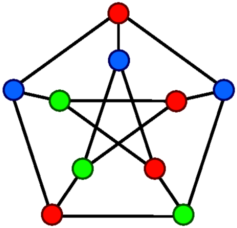

Figure : Petersen Graph with a 3-coloring. $\chi(\text{Petersen}) = 3$.

Simplest cases: Graphs with $\chi(G) = 1$ and $\chi(G) = 2$

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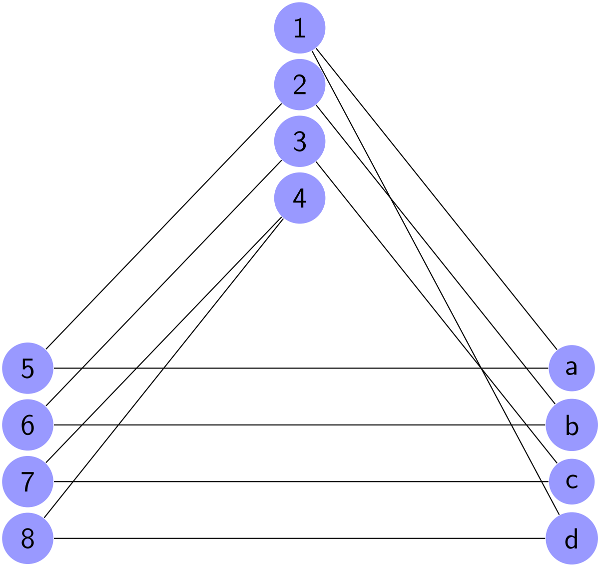
- If $\chi(G) = 1$ then G has no edges.
- If $\chi(G) = 2$ then G is non-trivial *bipartite*.
- Bad news: No 'nice' characterization for graphs of chromatic number k for any $k \geq 3$.

Why no nice characterization?



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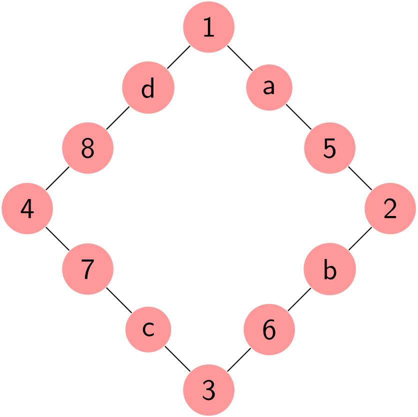

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- Proposition
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- Theorem
  (Brooks): If $G \neq C_{2n+1}, K_n$ *and is connected then* $\chi(G) \leq \Delta$.

Lower bounds

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- $\chi(G) \geq \frac{n}{\alpha(G)}$, where $\alpha(G) =$ Size of a maximum independent set in G.

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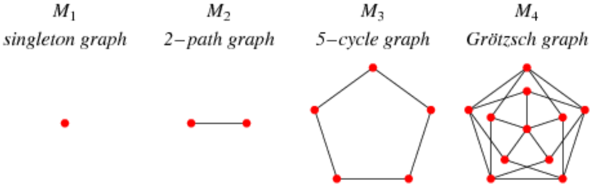

Figure : The Mycielski construction for $\chi(G) = 1, 2, 3, 4$.

Graphs with no small cycles and large chromatic number

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(Erdős) For any given k, g there exists a graph G with girth greater than g and $\chi(G) \geq k$.

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- In particular, if $\theta < 1/g$ we have $\mathbb{E}(N) = o(n)$, so $\mathbb{P}(N > n/2) < 0.1$, say.

Sketch of proof of Erdős' result (contd.)

- $$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left(ne^{-(p(x-1)/2}\right)^{x} < 0.1,$$

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- Delete from each small cycle an edge to destroy all cycles of size at most g (deleting at most $n/2$ vertices). The resulting graph G^* has $\alpha(G^*) < Cn^{1-\theta}\log n$ and has no cycles of size less than or equal to g. Furthermore, $\chi(G) \geq \chi(G^*) \geq \frac{n/2}{Cn^{1-\theta}\log n}$. ■

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To have witnessed such graphs, for $k = 6, g = 6$, one would have to consider $n \geq 2^{42}$ (!) This explains why it seemed 'counter-intuitive' that large chromatic number and large girth cannot happen simultaneously.

$\chi(G)$ and local considerations

Question: If we knew the chromatic number of every 'large' subgraph of G, then can we deduce something about $\chi(G)$? Can $\chi(G)$ still be much larger?

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- Proof uses a probabilistic construction.
- Almost every graph (in the random graph model) can be altered mildly to obtain such a G.

Any improvements on Brooks' theorem?

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All these proofs heavily rely on probabilistic techniques.

Some Open problems

1. (Hadwiger's conjecture) Let $\mathcal{G}$ be a class of graphs closed under deletions of vertices/edges, and contractions of edges, and removing any loops that might arise. Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices in a largest clique that occurs in $\mathcal{G}$.

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2. (B. Reed) $\chi(G) \leq \lceil \frac{\Delta+\omega+1}{2} \rceil$, where $\omega = \omega(G)$ is the size of a maximum clique in G.