Projective geometry for Computer Vision

Subhashis Banerjee

Department of Computer Science and Engineering
IIT Delhi

NIT, Rourkela
March 27, 2010
Overview

- Pin-hole camera
- Why projective geometry?
- Reconstruction
Correspondence problem: Match image projections of a 3D configuration.

Reconstruction problem: Recover the structure of the 3D configuration from image projections.

Re-projection problem: Is a novel view of a 3D configuration consistent with other views? (Novel view generation)
An infinitely strange perspective

- Parallel lines in 3D space converge in images.
- The line of the horizon is formed by ‘infinitely’ distant points (vanishing points).
- Any pair of parallel lines meet at a point on the horizon corresponding to their common direction.
- All ‘intersections at infinity’ stay constant as the observer moves.

Figure 1.1: Landscape with horizon
3D reconstruction from pin-hole projections

La Flagellazione di Cristo (1460) Galleria Nazionale delle Marche by Piero della Francesca (1416-1492) (Robotics Research Group, Oxford University, 2000)
The effects can be modelled mathematically using the ‘linear perspective’ or a ‘pin-hole camera’ (realized first by Leonardo?)

If the world coordinates of a point are \((X, Y, Z)\) and the image coordinates are \((x, y)\), then

\[ x = \frac{fX}{Z} \quad \text{and} \quad y = \frac{fY}{Z} \]

The model is non-linear.
In terms of projective coordinates

\[ \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \]

where, \( \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathcal{P}^2 \) and \( \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathcal{P}^3 \) are **homogeneous coordinates**.
Euclidean and Affine geometries

- Given a coordinate system, $n$-dimensional real affine space is the set of all points parameterized by $\mathbf{x} = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$.

- An affine transformation is expressed as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}$$

where $A$ is a $n \times n$ (usually) non-singular matrix and $\mathbf{b}$ is a $n \times 1$ vector representing a translation.

- By SVD

$$A = U\Sigma V^T = (UV^T)(V\Sigma V^T) = R(\theta)R(-\phi)\Sigma R(\phi)$$

where

$$\Sigma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$
Euclidean and Affine geometries

▶ In the special case of when $A$ is a rotation (i.e., $AA^t = A^t A = I$), then the transformation is *Euclidean*.
▶ An affine transformation preserves parallelism and ratios of lengths along parallel directions.
▶ An Euclidean transformation, in addition to the above, also preserves lengths and angles.
▶ *Since an affine (or Euclidean) transformation preserves parallelism it cannot be used to describe a pinhole projection.*
Spherical geometry

The space $S^2$:

$$S^2 = \{ \mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x}|| = 1 \}$$

lines in $S^2$: Viewed as a set in $\mathbb{R}^3$ this is the intersection of $S^2$ with a plane through the origin. We will call this great circle a line in $S^2$. Let $\xi$ be a unit vector. Then, $l = \{ \mathbf{x} \in S^2 : \xi^t \mathbf{x} = 0 \}$ is the line with pole $\xi$. 

\[ \xi \]
Spherical geometry

- Lines in $S^2$ cannot be parallel. Any two lines intersect at a pair of antipodal points.

- A point on a line:
  \[ \mathbf{l} \cdot \mathbf{x} = 0 \text{ or } \mathbf{l}^T \mathbf{x} = 0 \text{ or } \mathbf{x}^T \mathbf{l} = 0 \]

- Two points define a line:
  \[ \mathbf{l} = \mathbf{p} \times \mathbf{q} \]

- Two lines define a point:
  \[ \mathbf{x} = \mathbf{l} \times \mathbf{m} \]
The projective plane $\mathcal{P}^2$ is the set of all pairs $\{x, -x\}$ of antipodal points in $S^2$.

Two alternative definitions of $\mathcal{P}^2$, equivalent to the preceding one are

1. The set of all lines through the origin in $\mathbb{R}^3$.
2. The set of all equivalence classes of ordered triples $(x_1, x_2, x_3)$ of numbers (i.e., vectors in $\mathbb{R}^3$) not all zero, where two vectors are equivalent if they are proportional.
The space $\mathcal{P}^2$ can be thought of as the infinite plane tangent to the space $S^2$ and passing through the point $(0, 0, 1)^t$. 
Let $\pi : S^2 \rightarrow \mathcal{P}^2$ be the mapping that sends $x$ to $\{x, -x\}$. The $\pi$ is a two-to-one map of $S^2$ onto $\mathcal{P}^2$.

A line of $\mathcal{P}^2$ is a set of the form $\pi l$, where $l$ is a line of $S^2$. Clearly, $\pi x$ lies on $\pi l$ if and only if $\xi^t x = 0$.

**Homogeneous coordinates:** In general, points of real $n$-dimensional **projective space**, $\mathcal{P}^n$, are represented by $n + 1$ component column vectors $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ such that at least one $x_i$ is non-zero and $(x_1, \ldots, x_n, x_{n+1})$ and $(\lambda x_1, \ldots, \lambda x_n, \lambda x_{n+1})$ represent the same point of $\mathcal{P}^n$ for all $\lambda \neq 0$.

$(x_1, \ldots, x_n, x_{n+1})$ is the homogeneous representation of a projective point.
Canonical injection of $\mathbb{R}^n$ into $\mathcal{P}^n$

- Affine space $\mathbb{R}^n$ can be embedded in $\mathcal{P}^n$ by
  \[(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, 1)\]

- Affine points can be recovered from projective points with $x_{n+1} \neq 0$ by
  \[(x_1, \ldots, x_n) \sim \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right) \rightarrow \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)\]

- A projective point with $x_{n+1} = 0$ corresponds to a point at infinity.

- The ray $(x_1, \ldots, x_n, 0)$ can be viewed as an additional ideal point as $(x_1, \ldots, x_n)$ recedes to infinity in a certain direction. For example, in $\mathcal{P}^2$,
  \[
  \lim_{T \to 0} \left(\frac{X}{T}, \frac{Y}{T}, 1\right) = \lim_{T \to 0} (X, Y, T) = (X, Y, 0)
  \]
Lines in $\mathcal{P}^2$

- A line equation in $\mathbb{R}^2$ is
  \[ a_1 x_1 + a_2 x_2 + a_3 = 0 \]

- Substituting by homogeneous coordinates $x_i = X_i/X_3$ we get a homogeneous linear equation
  \[(a_1, a_2, a_3) \cdot (X_1, X_2, X_3) = \sum_{i=1}^{3} a_i X_i = 0, \quad \mathbf{X} \in \mathcal{P}^2 \]

- A line in $\mathcal{P}^2$ is represented by a homogeneous 3-vector $(a_1, a_2, a_3)$.
- A point on a line: $\mathbf{a} \cdot \mathbf{X} = 0$ or $\mathbf{a}^T \mathbf{X} = 0$ or $\mathbf{X}^T \mathbf{a} = 0$
- Two points define a line: $\mathbf{l} = \mathbf{p} \times \mathbf{q}$
- Two lines define a point: $\mathbf{x} = \mathbf{l} \times \mathbf{m}$
The line at infinity

- The line at infinity \((l_\infty)\): is the line of equation \(X_3 = 0\). Thus, the homogeneous representation of \(l_\infty\) is \((0, 0, 1)\).
- The line \((u_1, u_2, u_3)\) intersects \(l_\infty\) at the point \((-u_2, u_1, 0)\).
- Points on \(l_\infty\) are directions of affine lines in the embedded affine space (can be extended to higher dimensions).
Conics in $\mathcal{P}^2$

A **conic** in affine space (inhomogeneous coordinates) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Homogenizing this by replacements $x = X_1/X_3$ and $y = Y_1/Y_3$, we obtain

$$aX_1^2 + bX_1X_2 + cX_2^2 + dX_1X_3 + eX_2X_3 + fX_3^2 = 0$$

which can be written in matrix notation as $\mathbf{X}^T \mathbf{C} \mathbf{X} = 0$ where $\mathbf{C}$ is symmetric and is the **homogeneous representation** of a **conic**.
The line $l$ tangent to a conic $C$ at any point $x$ is given by $l = Cx$.

$x^tCx = 0 \implies (C^{-1}l)^tC((C^{-1}l) = l^tC^{-1}l = 0$
(because $C^{-t} = C^{-1}$). This is the equation of the dual conic.
The degenerate conic of rank 2 is defined by two line $l$ and $m$ as
\[ C = lm^t + ml^t \]
Points on line $l$ satisfy $l^tx = 0$ and are hence on the conic because $(x^tl)(m^tx) + (x^tm)(l^tx) = 0$. (Similarly for $m$).
The dual conic $xy^t + yx^t$ represents lines passing through $x$ and $y$. 
Projective basis: A **projective basis** for $\mathcal{P}^n$ is any set of $n + 2$ points no $n + 1$ of which are linearly dependent.

**Canonical basis:**

- Points at infinity along each axis:
  \[
  \begin{bmatrix}
  1 \\
  0 \\
  \vdots \\
  0
  \end{bmatrix}, \quad
  \begin{bmatrix}
  0 \\
  1 \\
  \vdots \\
  0
  \end{bmatrix}, \ldots
  \]

- Origin:
  \[
  \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  1
  \end{bmatrix}
  \]

- Unit point:
  \[
  \begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1
  \end{bmatrix}
  \]
Change of basis: Let $e_1, e_2, \ldots, e_{n+1}, e_{n+2}$ be the standard basis and $a_1, a_2, \ldots, a_{n+1}, a_{n+2}$ be any other basis. There exists a non-singular transformation $[T]_{(n+1) \times (n+1)}$ such that:

$$T e_i = \lambda_i a_i, \forall i = 1, 2, \ldots, n + 2$$

$T$ is unique up to a scale.
Homography

The invertible transformation $T: \mathcal{P}^n \to \mathcal{P}^n$ is called a **projective transformation** or **collineation** or **homography** or **perspectivity** and is completely determined by $n + 2$ point correspondences.

- Preserves straight lines and cross ratios
- Given four collinear points $A_1, A_2, A_3$ and $A_4$, their **cross ratio** is defined as
  \[
  \frac{A_1A_3}{A_1A_4} \div \frac{A_2A_4}{A_2A_3}
  \]
- If $A_4$ is a point at infinity then the cross ratio is given as
  \[
  \frac{A_1A_3}{A_2A_3}
  \]
- The cross ratio is independent of the choice of the projective coordinate system.
Homography

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Homography

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If the points $x_i$ lie on the line $l$, we have $l^T x_i = 0$.

Since, $l^T H^{-1} H x_i = 0$ the points $H x_i$ all lie on the line $H^{-T} l$.

Hence, if points are transformed as $x'_i = H x_i$, lines are transformed as $l' = H^{-T} l$. 

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Note that a conic is represented (homogeneously) as

\[ x^T C x = 0 \]

Under a point transformation \( x' = Hx \) the conic becomes

\[ x^T C x = x'^T [H^{-1}]^T C H^{-1} x' = x'^T H^{-T} C H^{-1} x' = 0 \]

This is the quadratic form of \( x'^T C' x' \) with \( C' = H^{-T} C H^{-1} \). This gives the transformation rule for a conic.
The affine subgroup

In an affine space $\mathcal{A}^n$ an **affine transformation** defines a correspondence $\mathbf{X} \leftrightarrow \mathbf{X'}$ given by:

$$
\mathbf{X'} = \mathbf{A} \mathbf{X} + \mathbf{b}
$$

where $\mathbf{X}$, $\mathbf{X'}$ and $\mathbf{b}$ are $n$-vectors, and $\mathbf{A}$ is an $n \times n$ matrix. Clearly this is a subgroup of the projective group. Its projective representation is

$$
\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{c} \\ \mathbf{0}_n^T & t_{33} \end{bmatrix}
$$

where $\mathbf{A} = \frac{1}{t_{33}} \mathbf{C}$ and $\mathbf{b} = \frac{1}{t_{33}} \mathbf{c}$. 

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Affine transformations preserve the plane/line at infinity

\[
\begin{bmatrix}
A & b & 0^t \\
0 & 1 & 1
\end{bmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
A \\
0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
0
\end{pmatrix}
\]

A general projective transformation moves points at infinity to finite points.
The Euclidean subgroup

- The absolute conic: The conic $\Omega_\infty$ is intersection of the quadric of equation:

$$\sum_{i=1}^{n+1} x_i^2 = x_{n+1} = 0 \text{ with } \pi_\infty$$

- In a metric frame $\pi_\infty = (0, 0, 0, 1)^T$, and points on $\Omega_\infty$ satisfy

$$X_1^2 + X_2^2 + X_3^3 + X_4 = 0$$

- For directions on $\pi_\infty$ (with $X_4 = 0$), the absolute conic $\Omega_\infty$ can be expressed as

$$(X_1, X_2, X_3)I(X_1, X_2, X_3)^T = 0$$

- The absolute conic, $\Omega_\infty$, is fixed under a projective transformation $H$ if and only if $H$ is an Euclidean transformation.
Affine calibration of a plane
Affine calibration of a plane

If the imaged line at infinity is \( l = (l_1, l_2, l_3)^t \), then provided \( l_3 \neq 0 \) a suitable projective transformation that maps \( l \) back to \( l_\infty \) is

\[
H = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
l_1 & l_2 & l_3
\end{bmatrix} H_A
\]
Reconstruction

Camera recovery

Metrology
Surfaces of revolution

Axis of Revolution

Silhouette

Tangent Plane

C
Modeling of structured scenes

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Projective geometry for Computer Vision
A walkthrough