Duality Transformation and its Application to Computational Geometry

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Outline

Introduction

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- In this lecture we explore how geometric duality can be used to design efficient algorithms by considering a number of problems in computational geometry.
- For simplicity, we consider duality in two dimensions only. However, the concept generalizes to higher dimensions also.

Introduction

- In the Cartesian plane, a point has two parameters (x- and y-coordinates) and a (non-vertical) line also has two parameters (slope and y-intercept). We can thus map a set of points to a set of lines, and vice versa, in an one-to-one manner.

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- This natural duality between points and lines in the Cartesian plane has long been known to geometers.

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- Each such mapping has its advantages and disadvantages in particular contexts.

Outline

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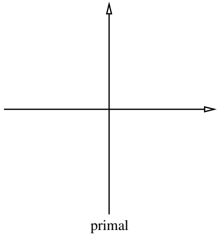

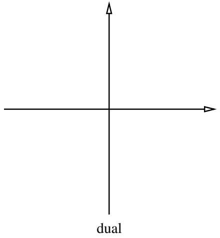

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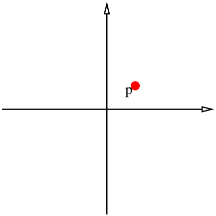
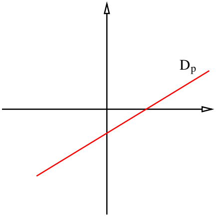


primal



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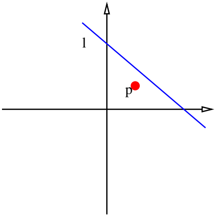

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Definition

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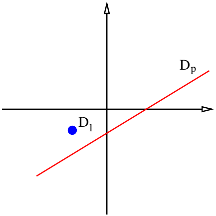

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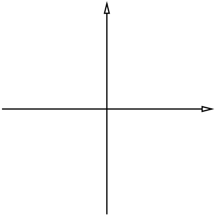
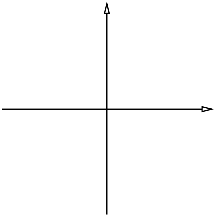
However this is not a problem in general. Because we can always rotate the problem space slightly so that no line is vertical. Sometimes, vertical lines are taken as special cases and treated separately.

Properties

Incidence is preserved



primal



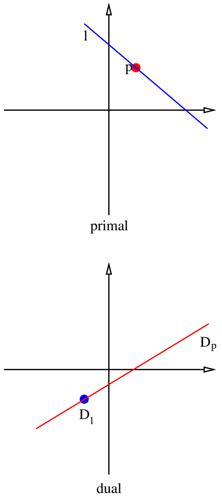
dual

Properties

Incidence is preserved

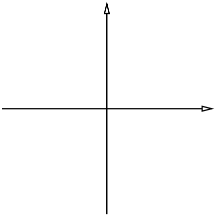
Lemma

A point $p(a,b)$ is incident to the line $l(y = cx + d)$ in the primal plane iff point $D_l(c, -d)$ is incident to the line $D_p(y = ax - b)$ in the dual plane.



Properties

But order is reversed



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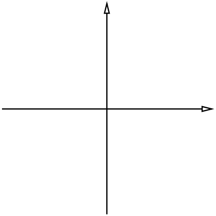

dual

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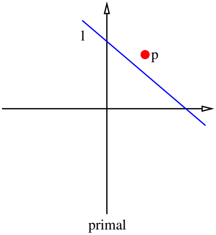
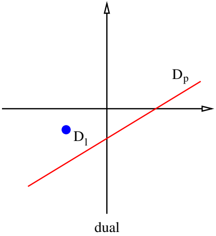
Lemma

A point $p(a, b)$ is above (below) the line $l(y = cx + d)$ in the primal plane iff line $D_p(y = ax - b)$ is below (above) the point $D_l(c, -d)$ in the dual plane.





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- An alternative definition, called polar duality, is also used.

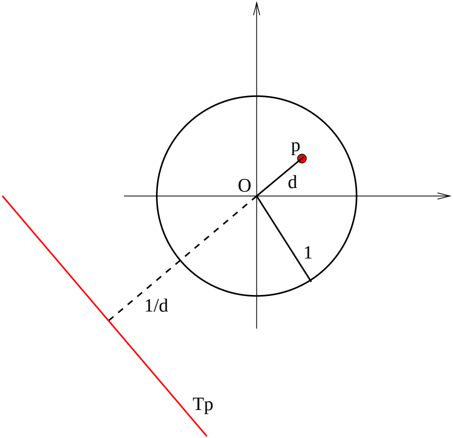
Polar Duality

Definition

A point p with coordinates (a, b) in the primal plane corresponds to a line T_p with equation $ax + by + 1 = 0$ in the dual plane and vice versa.

Polar Duality

- Geometrically this means that if d is the distance from the origin(O) to the point p, the dual T_p of p is the line perpendicular to Op at distance $1/d$ from O and placed on the other side of O.



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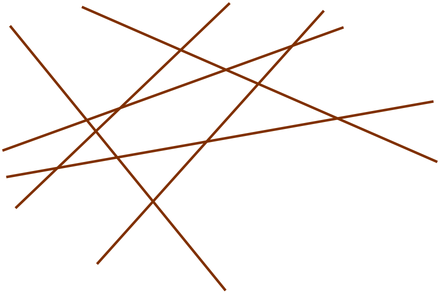
Let $\mathcal{L}$ be a set of n lines in the plane. The embedding of $\mathcal{L}$ in the plane induces a planner subdivision that consists of vertices, edges, and faces where some of the edges and faces are unbounded. This subdivision is referred to as arrangement induced by $\mathcal{L}$, and is denoted by $A(\mathcal{L})$.



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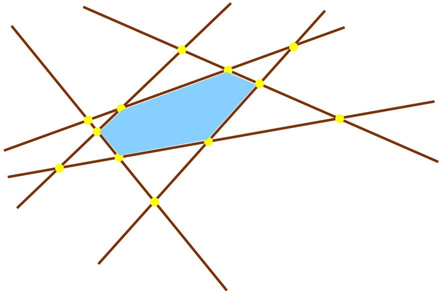

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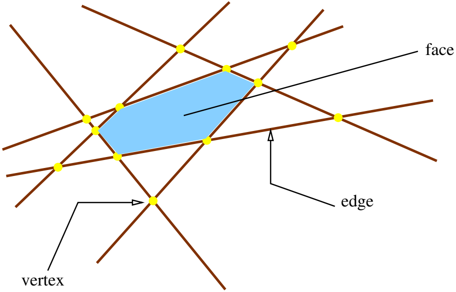

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Observation

Worst case complexity occurs when an arrangement is simple.

Result

Theorem

Let $\mathcal{L}$ be the set of n lines in the plane, and let $A(\mathcal{L})$ be the arrangement induced by $\mathcal{L}$.

(i) *The number of vertices of $A(\mathcal{L})$ is at most $n(n-1)/2$.*

(ii) *The number of edges of $A(\mathcal{L})$ is at most n^2.*

(iii) *The number of faces of $A(\mathcal{L})$ is at most $n^2/2 + n/2 + 1$.*

Equality holds in these three statements iff $A(\mathcal{L})$ is simple.

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Can be proved easily by using Euler's formula:
For any connected planner embedded graph with m_v vertices, m_e edges, and m_f faces the following relation holds

$$m_v - m_e + m_f = 2.$$

Computation of Arrangement

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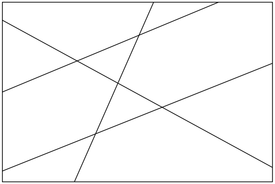
- Algorithms for a number of problems are based on constructing and analyzing the arrangement of a specific set of lines.
- A variety of data structures have been proposed for this purpose.

Computation of Arrangement

- Algorithms for constructing arrangements are usually incremental. Beginning with an arrangement of one or two lines, subsequent lines are inserted into the arrangement one at a time, thereby building larger and larger arrangements.



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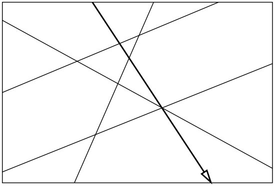
- To insert a new line, we start on the leftmost cell containing the line and walk over the arrangement to the right, moving from cell to neighboring cell and splitting into two pieces those cells that contain the new line.



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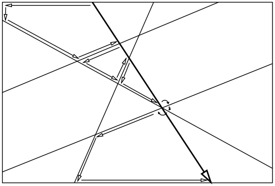

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- We can thus scan through each edge of every cell encountered on our insertion walk in linear time.
- The total time to insert all n lines in constructing the full arrangement is $O(n^2)$.

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Given a set $\mathcal{L}$ of n lines in the plane, the arrangement $A(\mathcal{L})$ induced by $\mathcal{L}$ can be constructed in $O(n^2)$ time.

Levels

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- It is simple both from understanding and implementations point of view.

Definition

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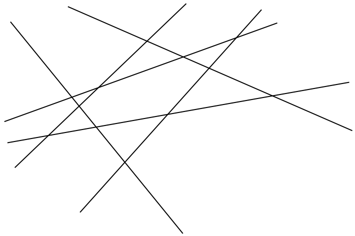
Let $\mathcal{L}$ be a set on n lines in the plane inducing an arrangement $A(\mathcal{L})$. A point π in the plane is at level θ ($0 \leq \theta \leq n$) if there are exactly θ lines in $\mathcal{L}$ that lie strictly below π. The θ-level of $A(\mathcal{L})$ is the closure of a set of points on the lines of $\mathcal{L}$ whose levels are exactly θ in $A(\mathcal{L})$, and is denoted as λ_θ.

Observations



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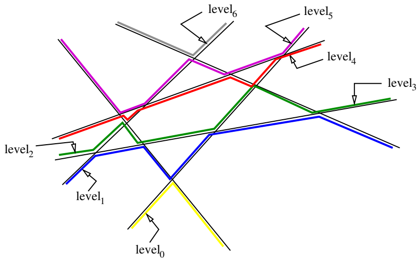
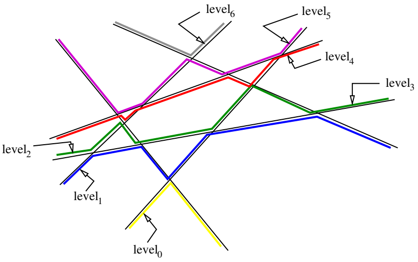


Observations

- Clearly, the edges of λ_θ form a monotone polychain from $x = -\infty$ to $x = \infty$. Each vertex of the arrangement $A(\mathcal{L})$ appears in two consecutive levels, and each edge of $A(\mathcal{L})$ appears in exactly one level.



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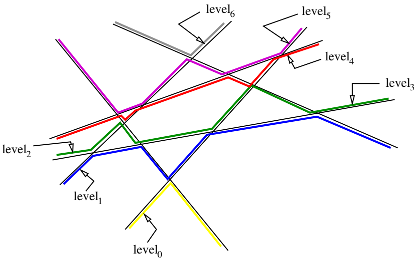
- We can thus store each level simply as an array of segments.



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- Here we consider an alternative method using plane sweep paradigm.
- The method was first introduced by Bentley and Ottmann (1979) in the context of solving the problem of line segment intersections.

Plane Sweep Method

Basic method consists of:

- A vertical line l, called the sweep line, sweeps over the arrangement from $x = -\infty$ to $x = \infty$. Observe that, at every instant, the sweep line intersects each element of $\mathcal{L}$.

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- The status of the sweep line at any instant is the order in which the lines intersect it.
- The status changes only when the sweep line crosses vertices of the arrangement which are intersection points of pairs of lines. These intersection points are called event points.
- The algorithm performs some computational steps when the sweep line reaches event points.

Data structure

- Apart from storing the events, we also need to insert new events and extract the event nearest to the sweep line on its right. Clearly, a queue is a suitable data structure for this.

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- We order the lines from bottom to top according to their intersections with the sweep line. Data structure we use for maintaining the sweep line status are arrays storing the levels. At an instant, portion of the line at the i-th position, $0 \leq i \leq n$, is part of the i-th level.

Processing

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- Portion of the line at the i-th position before the event point will become part of the $(i+1)$-th level after the event point. Similarly, portion of the line at the $(i+1)$-th position before the event point will become part of the i-th level after the event point.

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- If the line at the $(i+1)$-th position after the event point intersect the line at the $(i+2)$-th position on the right of the sweep line, then we insert the intersection point in the queue as a future event point. Similarly, if the line at the i-th position after the event point intersect the line at the $(i-1)$-th position on the right of the sweep line, then we insert this intersection point also as a future event point.

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- We then initialize the level arrays with the lines according to their position on the sweep line. This step needs $O(n)$ time.
- Finally, we check each pair of lines from bottom to top if they insert on the right of the sweep line. If yes, insert these intersection points in the queue as an event point. This step needs $O(n)$ time.

Algorithm

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  Process the event point.
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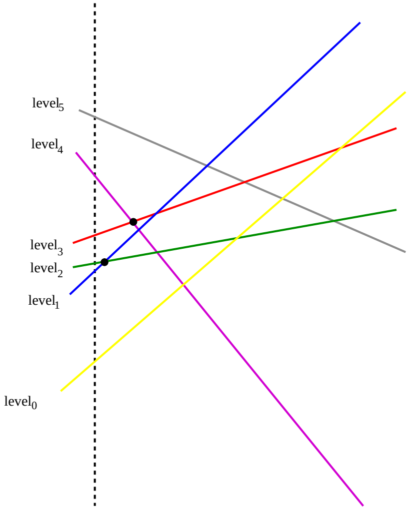
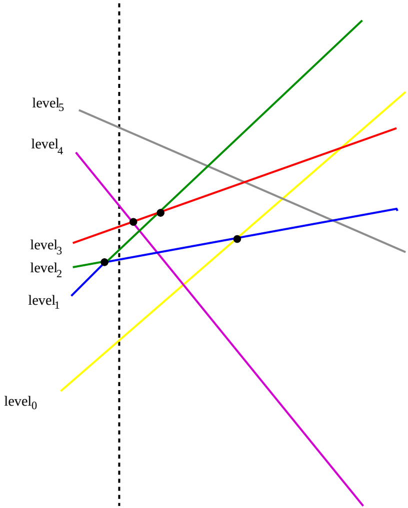
Example



level_5
level_4
level_3
level_2
level_1
level_0

Example



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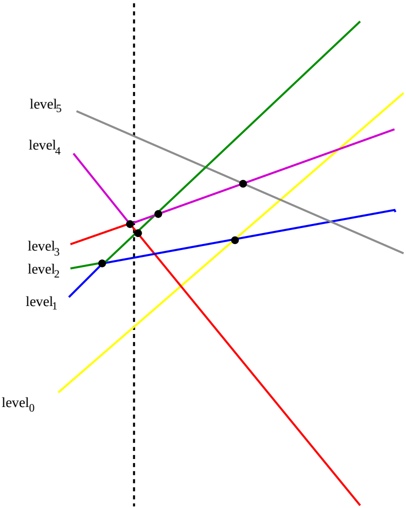

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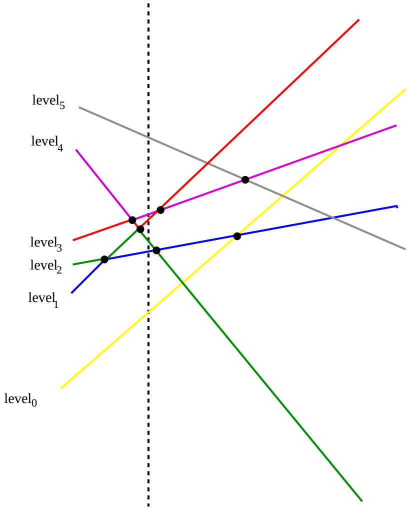

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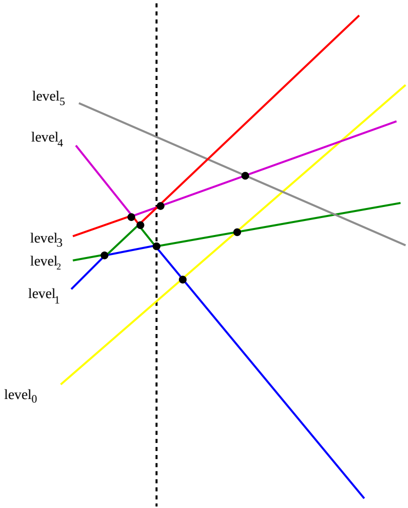
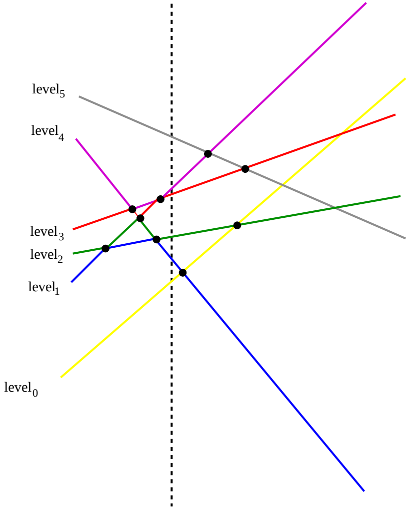
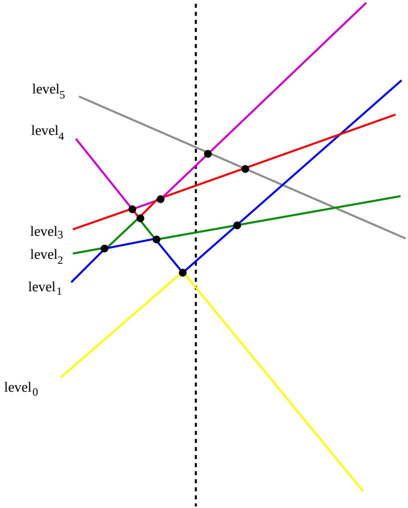


Example



level_5
level_4
level_3
level_2
level_1
level_0

Example



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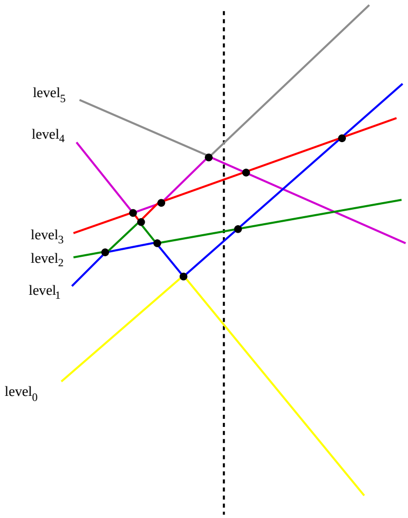

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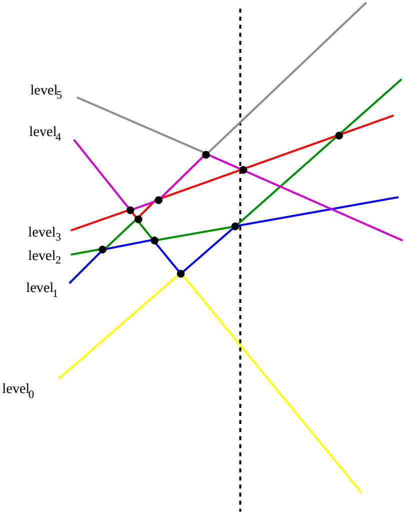

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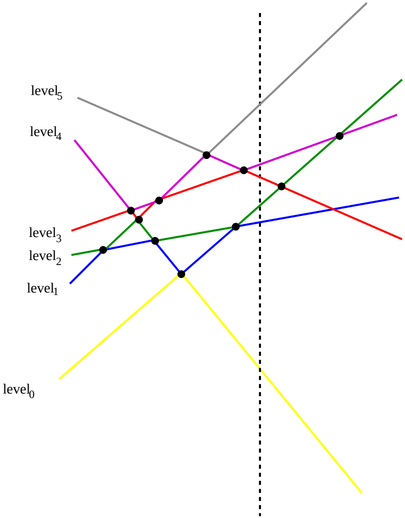

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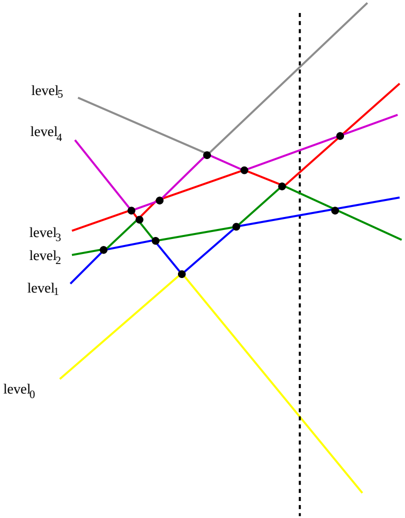

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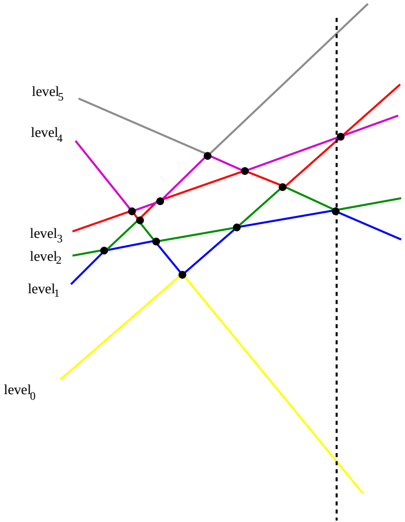

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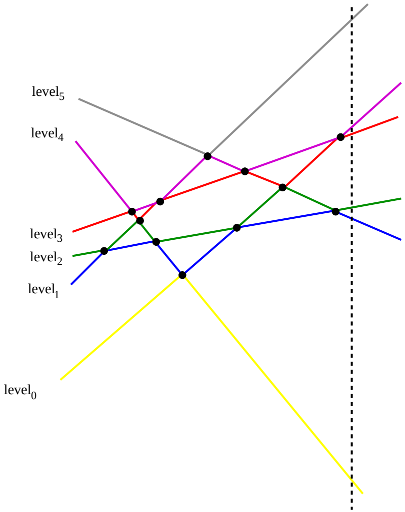

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- Space complexity is $O(n^2)$.

Result

Theorem

Using plane sweep, levels of an arrangement of n lines can be computed in $O(n^2 \log n)$ time using $O(n^2)$ space.

Outline

Smallest Area Triangle Problem

Problem

Let $\mathcal{P}$ be a set of n points in the plane. The problem is to determine which of the $\binom{n}{3}$ triangles with vertices in $\mathcal{P}$ has the smallest area.

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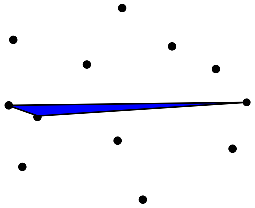
The solution of the above problem allows us to solve the following problem also.

Problem

Let $\mathcal{P}$ be a set of n points in the plane. The problem is to determine whether three points in $\mathcal{P}$ are collinear.

Smallest Area Triangle Problem

- The difficulty of the problem arises from the fact that the vertices of the smallest triangle can be arbitrarily apart (i.e., absence of locality).



Result

- The best known algorithm, without using duality, for this problem has time and space complexities $O(n^2 \log n)$ and $O(n)$ respectively.
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- The best known algorithm, without using duality, for this problem has time and space complexities $O(n^2 \log n)$ and $O(n)$ respectively.
  (Edelsbrunner and Welzl, 1982).
- Using duality, it is possible to improve upon the complexity.

Assumption

- The definition of duality implies that if two points p_i and p_j in the primal plane have same x-coordinate values, then corresponding duals D_{p_i} and D_{p_j} are parallel in the dual plane.

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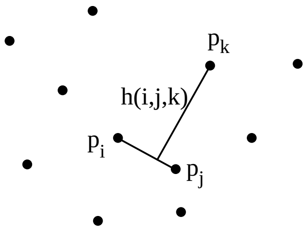
- To avoid this we assume that no two points in $\mathcal{P}$ have same x-coordinates. This may possibly require rotating the axes by a small angle which can be determined in $O(n \log n)$ time.

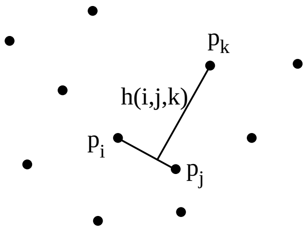
Sketch of the Solution

- Let $h(i, j, k)$ be the perpendicular distance from the point p_k to the segment $p_i p_j$.



Sketch of the Solution

- Let $h(i, j, k)$ be the perpendicular distance from the point p_k to the segment $p_i p_j$.
- Smallest area triangle with $p_i p_j$ as an edge minimizes $h(i, j, k)$ for all $k \neq i, j$; $1 \leq k \leq n$.



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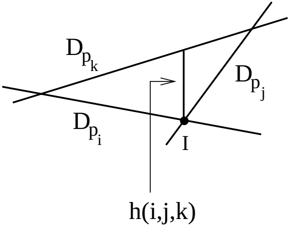
- However, when taken to dual plane, this leads to efficient algorithm.

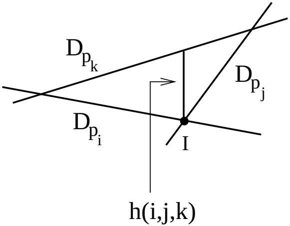
Dualization

- In the dual plane, the edge $p_i p_j$ becomes the intersection point I of D_{p_i} and D_{p_j}.



Dualization

- In the dual plane, the edge $p_i p_j$ becomes the intersection point I of D_{p_i} and D_{p_j}.
- The perpendicular from p_k on the edge $p_i p_j$ becomes vertical line segment from I to D_{p_k}.



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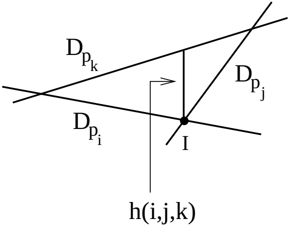
- Expression for the vertical distance $h(i,j,k)$ between I and D_{p_k} is



$$h(i,j,k) = y(p_k) + \frac{x(p_k)[y(p_j) - y(p_i)] + x(p_j)y(p_i) - x(p_i)y(p_j)}{x(p_i) - x(p_j)}$$

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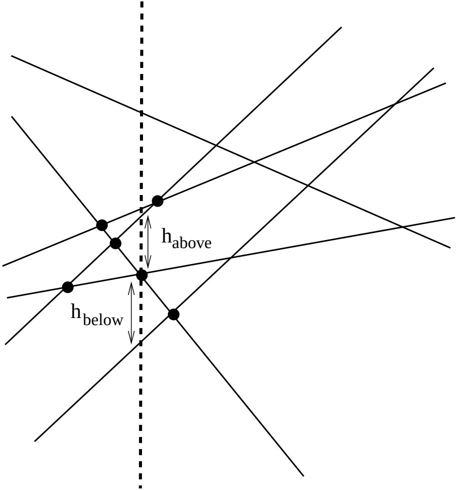
- Here event points are the intersection points between pairs of lines.

Algorithm

- When sweep line reaches an event point, the intersection point between D_{p_i} and D_{p_j} say, compute the vertical distances, along the sweep line, between the event point and the lines just above and below it.



Algorithm

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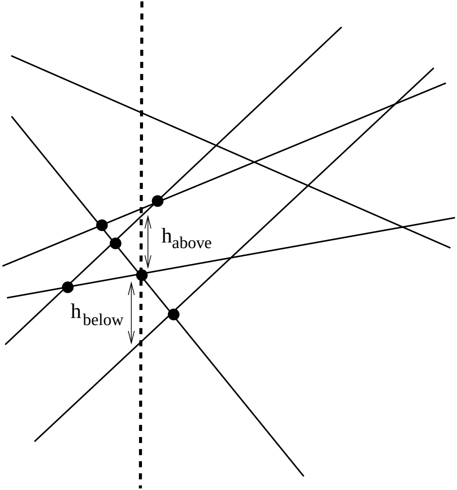
- Let the minimum distance occurs for the line D_{p_k}. Compute the minimum area of the triangle with $p_i p_j$ as base.



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- Hence space complexity of the algorithm is $O(n)$.
- Time complexity of the algorithm is, clearly, $O(n^2 \log n)$.
- The $\log n$ factor in the time complexity can be avoided by using topological line sweep.
  (Edelsbrunner, H. and Guibas, L. J., 1989)

Outline

Problem

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Given a set $\mathcal{P}$ of n points in the plane and a query line l, compute the nearest neighbor (in the perpendicular distance sense) of the query line l.

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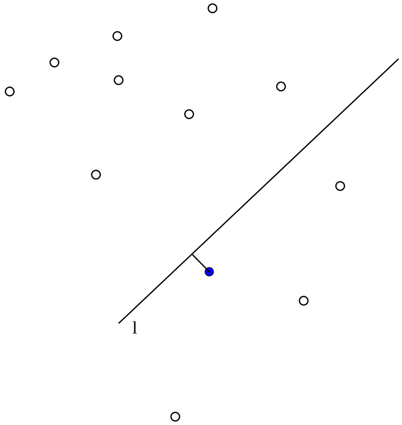

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- Here we are allowed to preprocess the point set so that each query can be answered efficiently.

Strategy

- We use duality to solve the problem.

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- Since our definition of duality does not allow vertical line, we need to have separate algorithm for handling vertical query lines.

Nearest Neighbor Query Vertical Line

- Sort the points of the given set $\mathcal{P}$ on their x-coordinates. This can be done in $O(n \log n)$ time using $O(n)$ space.

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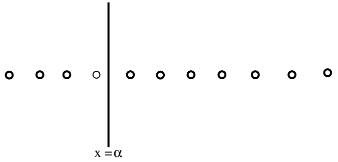

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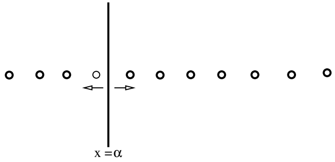

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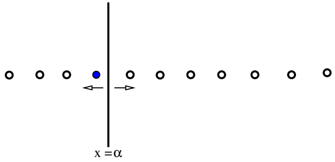
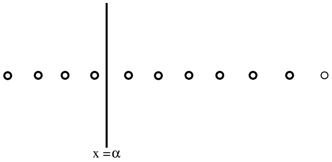


Farthest Neighbor Vertical Query Line

- Same scheme can also be used for determining the farthest neighbor of a query vertical line.



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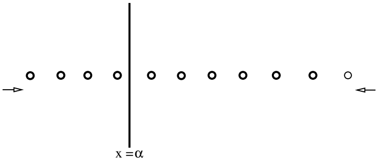

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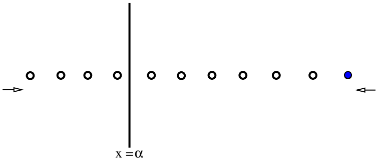

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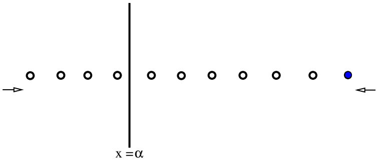


Result

Lemma

With $O(n \log n)$ preprocessing time using $O(n)$ space, nearest and farthest neighbors of a query vertical line can be found in $O(\log n)$ time.

Farthest Neighbor of a Non-Vertical Query Line

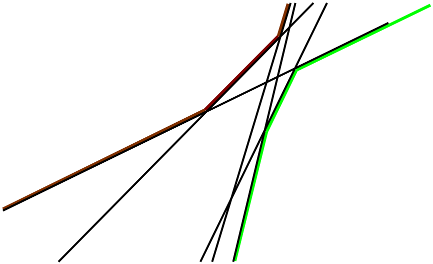
- Suppose the problem is to report the farthest neighbor of a given query line which is non-vertical.

Farthest Neighbor of a Non-Vertical Query Line

- Suppose the problem is to report the farthest neighbor of a given query line which is non-vertical.
- As the preprocessing step, compute the upper envelope and the lower envelope of the set of lines dual to the given set of points $\mathcal{P}$. This can be done in in $O(n \log n)$ time using $O(n)$ space as mentioned previously.

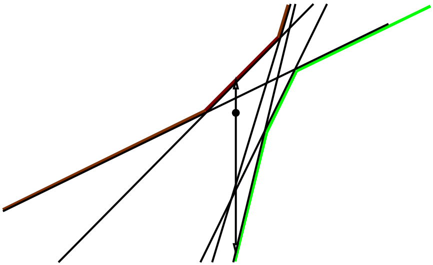
Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.



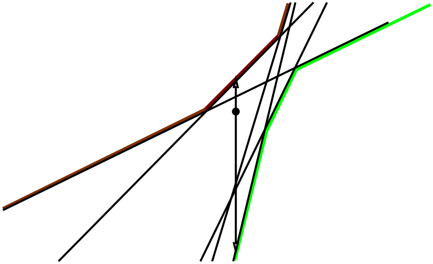
Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.
- Given a query line l, shoot a vertical ray from the point D_l in upward and downward direction and find the intersection points with the upper and the lower envelope respectively.



Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.
- Given a query line l, shoot a vertical ray from the point D_l in upward and downward direction and find the intersection points with the upper and the lower envelope respectively.
- This can be done in $O(\log n)$ time by using two binary searches on the arrays E_u and E_l holding the envelopes.



Result

Lemma

With $O(n \log n)$ preprocessing time using $O(n)$ space, farthest neighbors of a query non-vertical line can be found in $O(\log n)$ time.

Nearest Neighbor of a Query Non-vertical Line

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Nearest Neighbor of a Query Non-vertical Line

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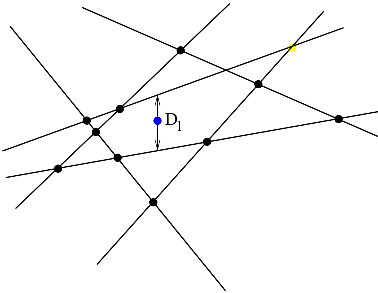
- Let $A(\mathcal{L})$ be the arrangement of lines of the set $\mathcal{L}$.
- Let f be the cell of the arrangement $A(\mathcal{L})$ containing D_l.
- Then one of the points corresponding to the lines just above D_l is the nearest neighbor of l in the primal plane.



Point Location Problem

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- So with standard data structure, nearest neighbors of a non-vertical query line can be determined in $O(\log n)$ time. The reqired preprocessing time and space is $O(n^2)$.

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- So with standard data structure, nearest neighbors of a non-vertical query line can be determined in $O(\log n)$ time. The reqired preprocessing time and space is $O(n^2)$.
- Here we describe an algorithm for point location using levels of arrangement.

Point Location Using Level Structure

- First compute the levels of the arrangement $A(\mathcal{L})$ in $O(n^2 \log n)$ time using $O(n^2)$ space.

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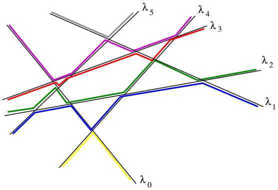
- Let λ_θ be the linear array containing vertices and edges of level θ, $\theta = 0, 1, \ldots, n$, of the arrangement $A(\mathcal{L})$.

Point Location Using Level Structure

- Create a balanced binary search tree T, called the primary structure, whose nodes correspond to the levels θ, $0 \leq \theta \leq n$. Each node of T, representing a level θ, is attached with the corresponding array λ_θ, called the secondary structure. This requires $O(n \log n)$ time and $O(n)$ space.



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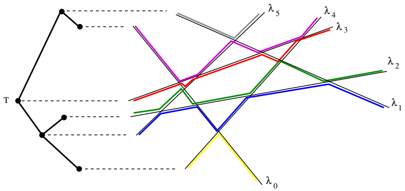
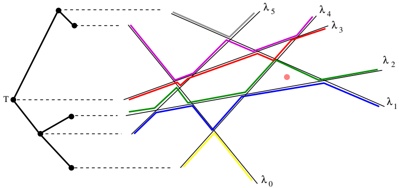


Point Location Using Level Structure

- Given the query line l, we perform two level binary search on the tree T with the point D_l.



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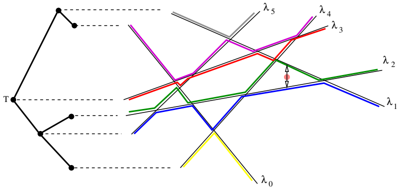

Point Location Using Level Structure

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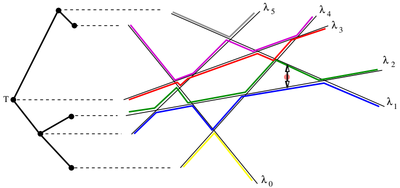
- Time complexity for performing this point location is $O(\log^2 n)$.



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Lemma

With $O(n^2 \log n)$ preprocessing time and $O(n^2)$ space, nearest neighbor of a non-vertical query line can be determined in $O(\log^2 n)$ time.

- It may be mentioned that the query time complexity can be reduced to $O(\log n)$, by using a data structuring technique, called fractional cascading.
  (Lueker, G. S., 1978)

Thank you!