Geometric Representations of Graphs

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- Conventionally graphs are represented as adjacency matrices, or adjacency lists. Algorithms are designed with such representations in mind usually.
- It is better to look at the structure of graphs and find some representations that are suitable for designing algorithms- say for a class of problems.
- Intersection graphs: The vertices correspond to the subsets of a set U. The vertices are made adjacent if and only if the corresponding subsets intersect.
- We propose to use some nice geometric objects as the subsets- like spheres, cubes, boxes etc. Here U will be the set of points in a low dimensional space.

- There are many situations where an intersection graph of geometric objects arises naturally....
- Some times otherwise NP-hard algorithmic problems become polytime solvable if we have geometric representation of the graph in a space of low dimension.

Boxicity and Cubicity

- **Cubicity:** Minimum dimension k such that G can be represented as the intersection graph of k-dimensional cubes.
- **Boxicity:** Minimum dimension k such that G can be represented as the intersection graph of k-dimensional axis parallel boxes.
- These concepts were introduced by F. S. Roberts, in 1969, motivated by some problems in ecology.
- By the later part of eighties, the research in this area had diminished.

An Equivalent Combinatorial Problem

- The boxicity(G) is the same as the minimum number k such that there exist interval graphs $I_1, I_2, \ldots, I_k$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_k$.
- Similarly, cubicity(G) is the minimum number k such that there exists unit interval graphs $I_1, \ldots, I_k$ such that $G = I_1 \cap \cdots \cap I_k$.

How to show a graph of high boxicity

- Let G be the complement of a perfect matching on n vertices. (Assume n is even).
- Suppose it is the intersection $n/2 - 1$ interval graphs.
- Then out of the $n/2$ missing edges of G, at least 2 should be missing in the same interval graph- by pigeon hole principle.
- Then it cannot be an interval graph, since there will be an induced 4 cycle !
- So, the boxicity of this graph is at least $n/2$.

A Simple Upper Bound

- Take a vertex u.
- Map u to the interval $[0, 1]$.
- Map each vertex in $N(u)$ to $[1, 2]$.
- Map each vertex in $V - (\{u\} \cup N(u))$ to $[2, 3]$.
- Do the same thing for each vertex u. We get n interval graphs.
- So, boxicity of $G \leq n$.

How to Improve the above strategy

- Can we deal with 2 vertices at a time ?
- What kind of pairs can be selected ? Roberts suggests to pick a pair of non-adjacent vertices.
- Let u and v be non-adjacent.
- Let u be given $[0, 1]$ and v be given $[4, 5]$.
- Remaining vertices belong to one of $S_0 = N(u) \cap N(v)$, $S_1 = N(u) - S_0$, $S_2 = N(v) - S_0$. $S_3 = V - (N(u) \cup N(v))$.
- To vertices of S_0 give $[1, 4]$
- To vertices of S_1 give $[1, 2.5]$
- To vertices of S_2 give $[2.5, 4]$
- To vertices of S_3 give $[2, 3]$
- Repeat the procedure. When do we get stuck ?
- This strategy gives an upper bound of $\lceil n/2 \rceil$

Boxicity and Maximum Degree

Boxicity of any graph is at most $2\Delta^2$, where Δ is the maximum degree of the graph.

(The only previous known upper bound was $n/2$ where n is the number of vertices)

Cubicity of any graph is $O(\Delta \log n)$, where Δ is the maximum degree and n is the number of vertices.

For the first time, we applied probabilistic tools in the study of boxicity and cubicity.

We related cubicity and boxicity with width parameters such as bandwidth and treewidth.

1. boxicity(G) $\leq$ treewidth(G) $+ 2$.

Treewidth is a very well studied parameter. This allowed us to get many results regarding boxicity.

The following consequence of the treewidth upper bound is interesting.

For a chordal graph, boxicity is at most $\chi(G) + 1$.

- cubicity$(G) \leq$ bandwidth$(G) + 1$.
- cubicity$(G) = O(b \log n)$, where b is the bandwidth and n is the number of vertices.

Another upper bound: boxicity $(G) \leq \lfloor \frac{t}{2} \rfloor + 1$, where t is the cardinality of the minimum vertex cover in G.

Some other results we obtained

- Cubicity of d-dimensional hypercubes is $\Theta(d/\log d)$.

The Claw Number: Let ψ be the largest integer such that there exists an induced star on $\psi + 1$ vertices in G. The ψ is called the claw number of G.

- Cubicity of an interval graph is $O(\log \psi)$.

Note that $\psi \leq \Delta$.

Let G be an AT-free graph. Then:

- boxicity$(G) \leq \chi(G)$.
- cubicity$(G) \leq$ box $(G).(\lceil \log \psi(G) \rceil + 2)$
- If girth of G is at least 5, then boxicity$(G) \leq 2$.

Lower bounds for boxicity: We came up with two general methods to derive lower bounds for boxicity. Applying these methods we could derive many results, some of which are listed below.

- The boxicity of almost all graphs is $\Omega(d_{av})$, where d_{av} is the average degree of the graph.
- If the minimum degree is δ, then boxicity is at least $\frac{n}{2(n-\delta-1)}$