Introduction to Computational Geometry

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Outline

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- People deal more with straight or flat objects (lines, line segments, polygons) or simple curved objects as circles, than with high degree algebraic curves.
- This branch of study is around thirty years old if one assumes Michael Ian Shamos's thesis [6] as the starting point.

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- Programming in CG is a little difficult. Fortunately, libraries like LEDA [7] and CGAL [8] are now available. These libraries implement various data structures and algorithms specific to CG.
- CG algorithms suffer from the curse of degeneracies. So, we would make certain simplifying assumptions at times like no three points are collinear, no four points are cocircular, etc.

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- First we consider some geometric primitives, that is, problems that arise frequently in computational geometry.

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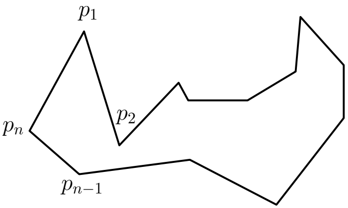
2. Area Computation of a Simple Polygon
3. Point Inclusion in a Simple Polygon
4. Convex Hull: An application of incremental algorithm
5. Art Gallery Problem: A study of combinatorial geometry

Area Computation

Problem

Given a simple polygon P of n vertices, compute its area.



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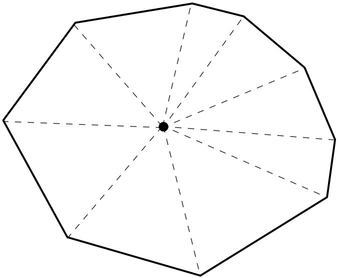
Find a point inside P, draw n triangles and compute the area.



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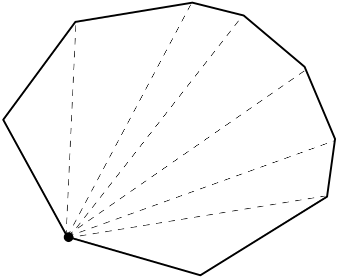
A better idea for convex polygon

We can triangulate P by non-crossing diagonals into $n-2$ triangles and then find the area.



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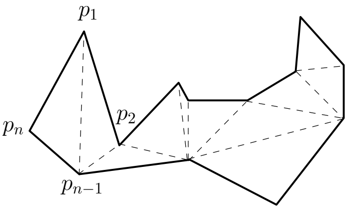
A better idea for simple polygon

We can do likewise.



Area Computation

Result

If P be a simple polygon with n vertices with coordinates of the vertex p_i being (x_i, y_i), $1 \leq i \leq n$, then twice the area of P is given by

$$2\mathcal{A}(P) = \sum_{i=1}^{n}(x_i y_{i+1} - y_i x_{i+1})$$

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Time complexity

We can triangulate P by a very complicated $O(n)$ algorithm [2] OR by a *more or less simple* $O(n \log n)$ time algorithm [1].

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Point Inclusion

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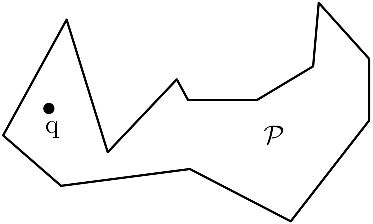
Given a simple polygon P of n points, and a query point q, is $q \in P$?



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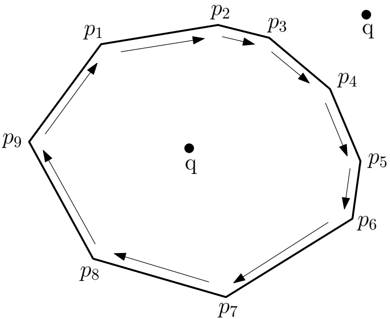
What if P is convex?

Easy in $O(n)$. Takes a little effort to do it in $O(\log n)$. Left as an exercise.



q is always to the right if $q \in \mathcal{P}$, else, it varies

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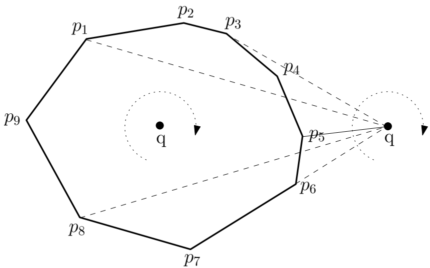
Another idea for convex polygon

Stand at q and walk around the polygon. We can show the same result for a simple polygon also.

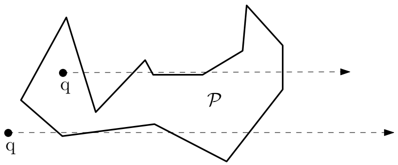


Total angular turn around q is 2π if $q \in \mathcal{P}$, else, 0

Point Inclusion

Another technique: Ray Shooting

Shoot a ray and count the number of crossings with edges of P. If it is odd, then $q \in P$. If it is even, then $q \notin P$. Some degenerate cases need to be handled. Time taken is $O(n)$.



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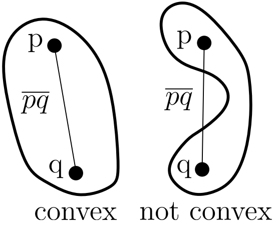

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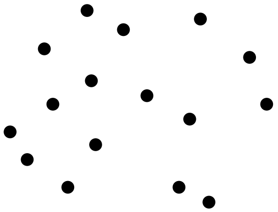

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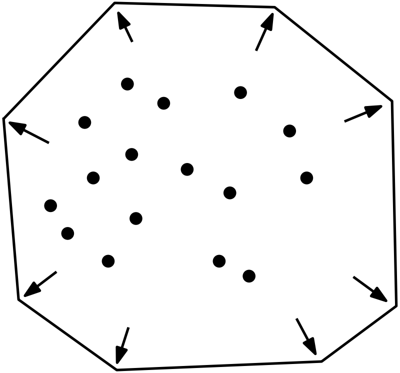

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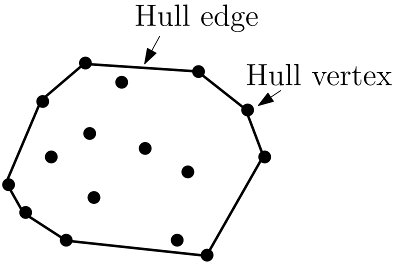

Convex Hull Problem

Problem

Given a set of points $\mathcal{P}$ in the plane, compute the convex hull $CH(\mathcal{P})$ of the set $\mathcal{P}$.

A Naive Algorithm

Outline

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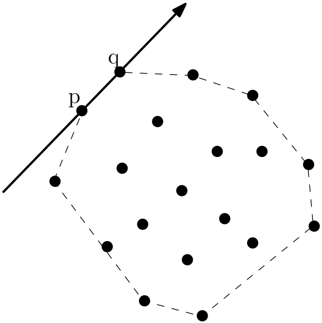

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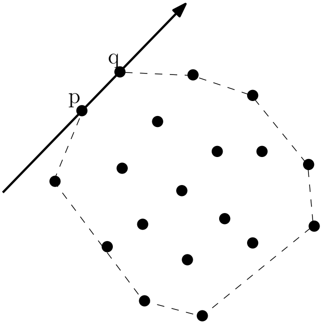
- If a line segment has all the other $n-2$ points on one side of it, then it is a hull edge.
- We need $\binom{n}{2}(n-2) = O(n^3)$ time.



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- Better characterizations lead to better algorithms.
- How much better can we make?
- Leads to the notion of lower bound of a problem.
- The problem of Convex Hull has a lower bound of $\Omega(n \log n)$. This can be shown by a reduction from the problem of sorting which also has a lower bound of $\Omega(n \log n)$.

Well Known Algorithms

- Grahams scan, time complexity $O(nlogn)$.
  (Graham, R.L., 1972)
- Divide and conquer algorithm, time complexity $O(nlogn)$.
  (Preparata, F. P. and Hong, S. J., 1977)
- Jarvis's march or gift wrapping algorithm, time complexity $O(nh)$ where h is the number of vertices of the convex hull.
  (Jarvis, R. A., 1973)
- Most efficient algorithm to date is based on the idea of Jarvis's march, time complexity $O(nlogh)$.
  (T. M. Chan, 1996)

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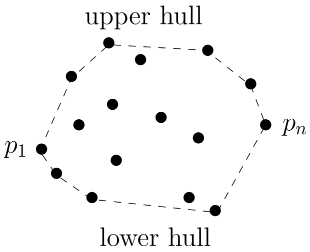
A better characterization

- Consider a walk in clockwise direction on the vertices of a closed polygon.



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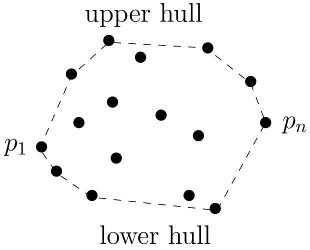

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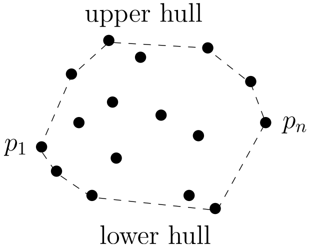

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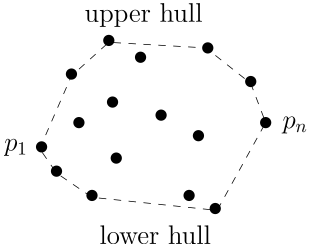

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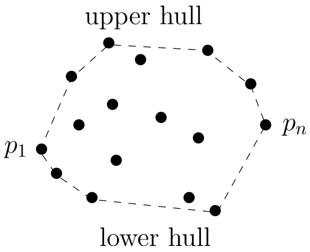

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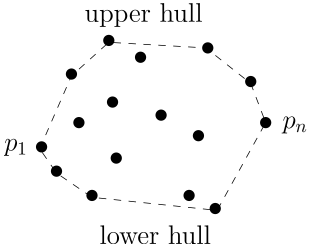
- Sort the points in $\mathcal{P}$ from left to right.



The Algorithm

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         Delete the middle of the last
         three points from L_U;
   }
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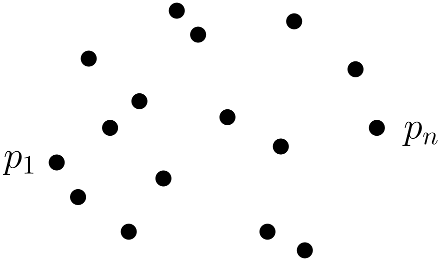
The Algorithm in Action



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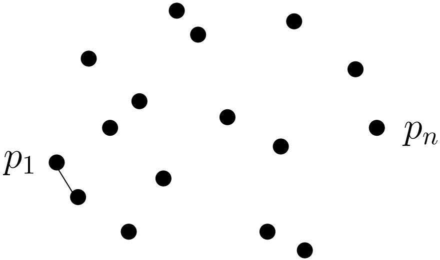

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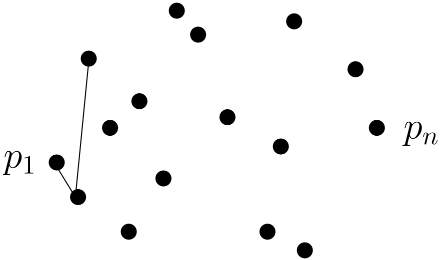

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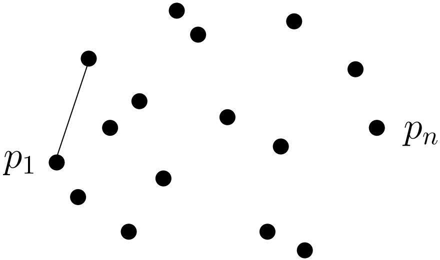

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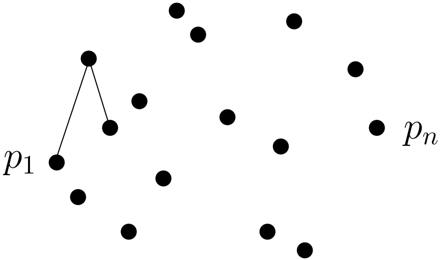

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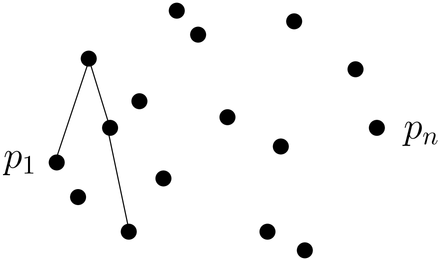

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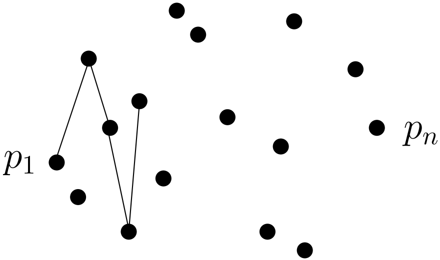

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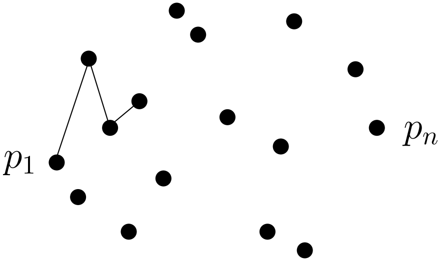

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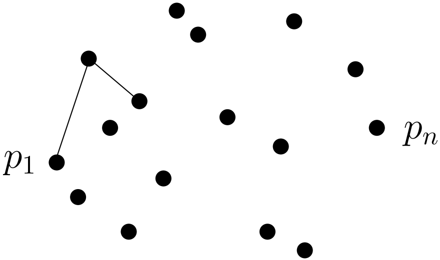

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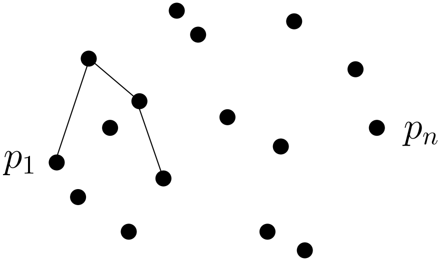

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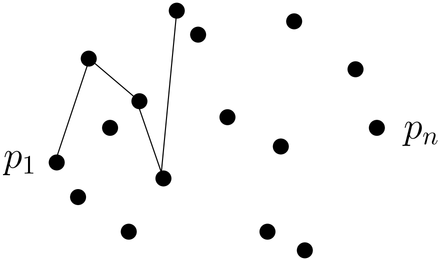

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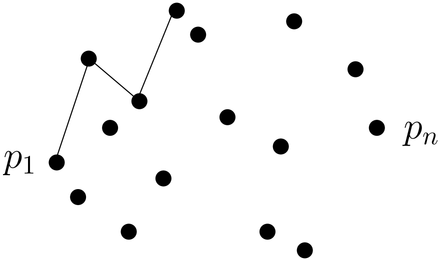

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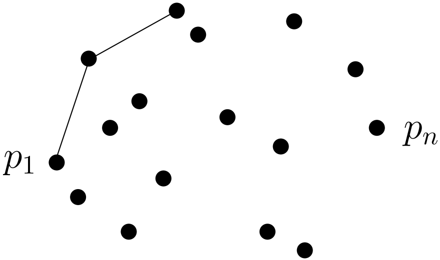

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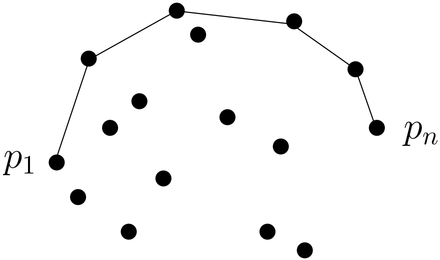

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- So, the total number of executions of the while loop body is bounded by $O(n)$.

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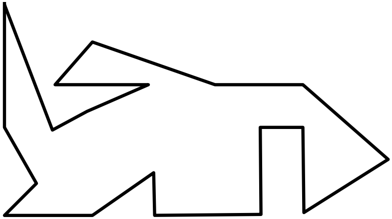
- Sorting takes time $O(n \log n)$.
- The for loop is executed $O(n)$ times.
- For each execution of the for loop, the while loop is encountered once.
- For each execution of the while loop body, a point gets deleted.
- A point once deleted, is never deleted again.
- So, the total number of executions of the while loop body is bounded by $O(n)$.
- Hence, the total time complexity is $O(n \log n)$.

Outline

Art Gallery Problem

The problem

Given a simple polygon $\mathcal{P}$ of n vertices, find the minimum number of cameras that can guard $\mathcal{P}$.



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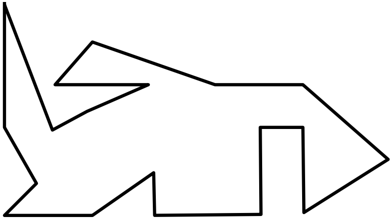
Hardness

The above problem is NP-Hard.



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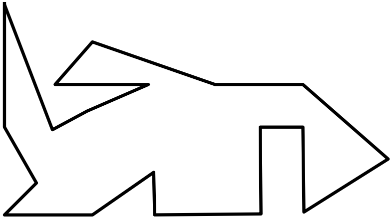

Art Gallery Problem

The problem

Given a simple polygon $\mathcal{P}$ of n vertices, find the minimum number of cameras that can guard $\mathcal{P}$.

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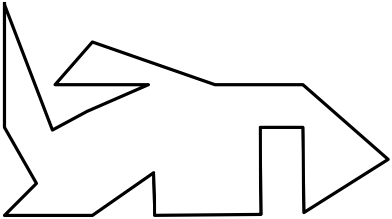
The above problem is NP-Hard.

Simpler Version

- Can we find, as a function of n, the number of cameras that suffices to guard $\mathcal{P}$?



Art Gallery Problem

The problem

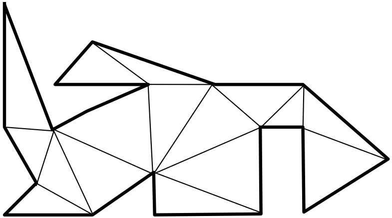
Given a simple polygon $\mathcal{P}$ of n vertices, find the minimum number of cameras that can guard $\mathcal{P}$.

Hardness

The above problem is NP-Hard.

Simpler Version

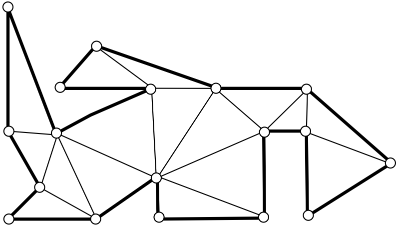
- Can we find, as a function of n, the number of cameras that suffices to guard $\mathcal{P}$?
- Recall $\mathcal{P}$ can be triangulated into $n-2$ triangles. Place a guard in each triangle.



Art Gallery Problem

Can the bound be reduced?

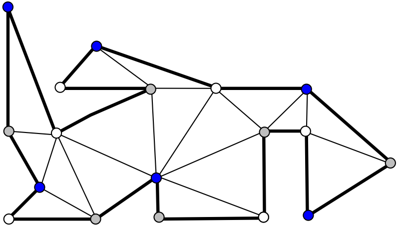
- Place guards at the vertices of the triangles.



Art Gallery Problem

Can the bound be reduced?

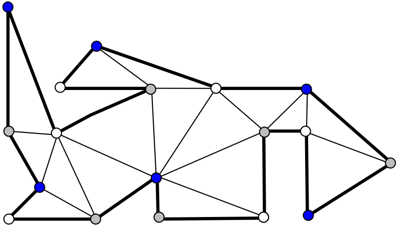
- Place guards at the vertices of the triangles.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.



Art Gallery Problem

Can the bound be reduced?

- Place guards at the vertices of the triangles.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.
- Choose the smallest color class to guard $\mathcal{P}$.



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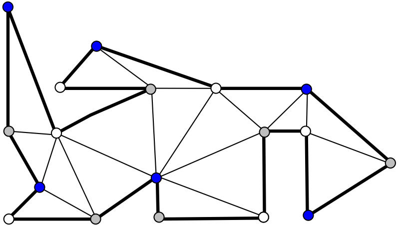
Can the bound be reduced?

- Place guards at the vertices of the triangles.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.
- Choose the smallest color class to guard $\mathcal{P}$.
- Hence, $\lfloor \frac{n}{3} \rfloor$ guards suffice.



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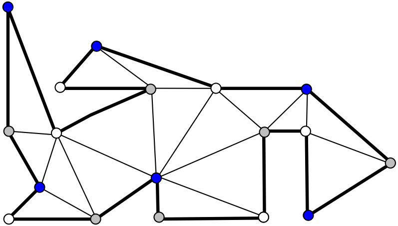
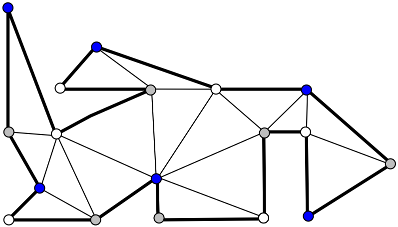
Can the bound be reduced?

- Place guards at the vertices of the triangles.
- We do a 3-coloring of the vertices of $\mathcal{T}$. Each triangle of $\mathcal{T}$ has a blue, gray and white vertex.
- Choose the smallest color class to guard $\mathcal{P}$.
- Hence, $\lfloor \frac{n}{3} \rfloor$ guards suffice.
- But, does a 3-coloring always exist?



Art Gallery Problem

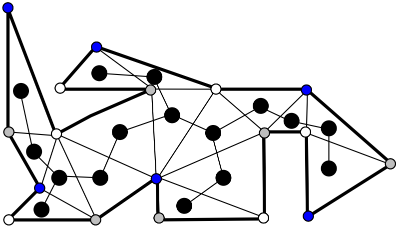
A 3-coloring always exist



Art Gallery Problem

A 3-coloring always exist

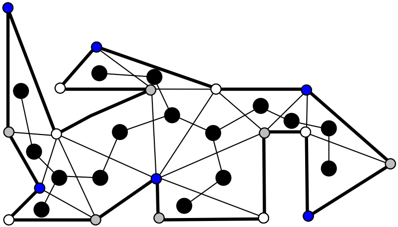
- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.



Art Gallery Problem

A 3-coloring always exist

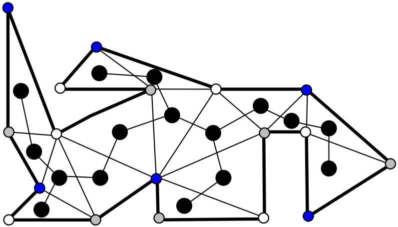
- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.



Art Gallery Problem

A 3-coloring always exist

- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.
- Do a DFS on $\mathcal{G}_{\mathcal{T}}$ to obtain the coloring.



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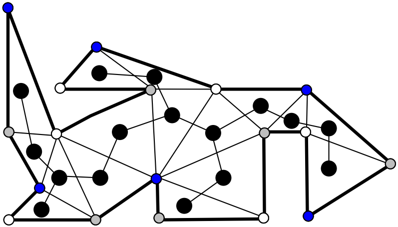
A 3-coloring always exist

- Consider the dual graph $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ of $\mathcal{P}$.
- $\mathcal{G}_{\mathcal{T}}$ is a tree as $\mathcal{P}$ has no holes.
- Do a DFS on $\mathcal{G}_{\mathcal{T}}$ to obtain the coloring.
- Place guards at those vertices that have color of the minimum color class. Hence, $\lfloor \frac{n}{3} \rfloor$ guards are sufficient to guard $\mathcal{P}$.



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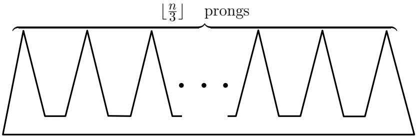
Necessity?

Are $\lfloor \frac{n}{3} \rfloor$ guards sometimes necessary?



Art Gallery Theorem

Final Result

For a simple polygon with n vertices, $\lfloor \frac{n}{3} \rfloor$ cameras are always sufficient and occasionally necessary to have every point in the polygon visible from at least one of the cameras.

References I

- Mark de Berg, Marc van Kreveld, Mark Overmars and Otfried Schwarzkof, *Computational Geometry: Algorithms and Applications*, Springer, 1997.
- B. Chazelle, *Triangulating a simple polygon in linear time*, Discrete Comput. Geom., 6:485524, 1991.
- Herbert Edelsbrunner, *Algorithms in Computational Geometry*, Springer, 1987.
- Joseph O'Rourke, *Computational Geometry in C*, Cambridge University Press, 1998.
- Franco P. Preparata and Michael Ian Shamos, *Computational Geometry: An Introduction*, Springer-Verlag, New York, 1985.
- Michael Ian Shamos, *Computational Geometry*, PhD thesis, Yale University, New Haven., 1978.

References II

- http://www.algorithmic-solutions.com
- http://www.cgal.org
- http://en.wikipedia.org/wiki/Computational_geometry

Thank you!