Graph Colorings

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An Information-theoretic Problem

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In this case, 1 bit will suffice.

What is Graph Coloring?

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Definition

Chromatic number of G: *The minimum k such that there is a k-coloring of G.*

The Chromatic number is denoted by $\chi(G)$.

Example: The Petersen Graph

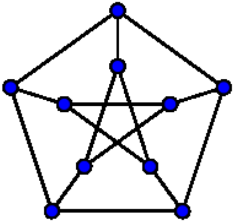

Figure: The Petersen Graph

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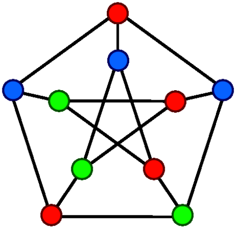

Figure: Petersen Graph with a 3-coloring.

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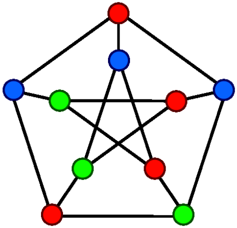

Figure: Petersen Graph with a 3-coloring. $\chi(\text{Petersen}) = 3$.

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To communicate the winner of any particular match, $\log_2 \chi(G)$ bits will suffice.

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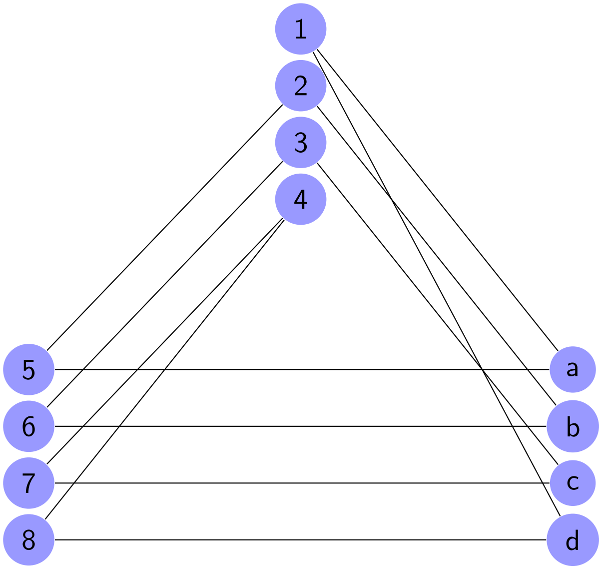
- If $\chi(G) = 1$ then G has no edges.
- If $\chi(G) = 2$ then G is non-trivial *bipartite*.
- Bad news: No 'nice' characterization for graphs of chromatic number k for any $k \geq 3$.

Why no nice characterization?



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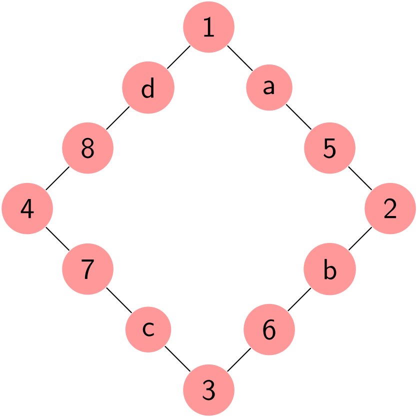

An upper bound from local considerations

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- Proposition
  $\chi(G) \leq \Delta + 1$, *where* $\Delta = \max_{v \in V} d(v)$.

- Theorem
  (Brooks): If $G \neq C_{2n+1}, K_n$ *and is connected then* $\chi(G) \leq \Delta$.

Lower bounds

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- $\chi(G) \geq \frac{n}{\alpha(G)}$, where $\alpha(G) =$ Size of a maximum independent set in G.

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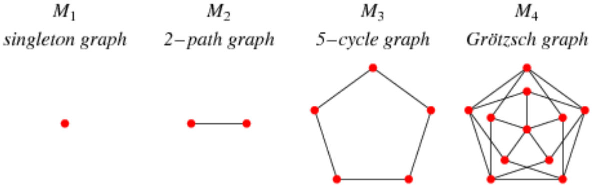

Figure: The Mycielski construction for $\chi(G) = 1, 2, 3, 4$.

Graphs with no small cycles and large chromatic number

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(Erdős) For any given k, g there exists a graph G with girth greater than g and $\chi(G) \geq k$.

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- If $N =$ number of cycles of size less than or equal to g, then $\mathbb{E}(N) = \sum_{i=3}^{g} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i < \frac{gn^{g\theta}}{6}$ if we have $p = n^{\theta - 1}$ (for some $0 < \theta < 1$).

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- In particular, if $\theta < 1/g$ we have $\mathbb{E}(N) = o(n)$, so $\mathbb{P}(N > n/2) < 0.1$, say.

Sketch of proof of Erdős' result (contd.)

- $$\mathbb{P}(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left(ne^{-(p(x-1)/2}\right)^{x} < 0.1,$$

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- Delete from each small cycle an edge to destroy all cycles of size at most g (deleting at most $n/2$ vertices). The resulting graph G^* has $\alpha(G^*) < Cn^{1-\theta}\log n$ and has no cycles of size less than or equal to g. Furthermore, $\chi(G) \geq \chi(G^*) \geq \frac{n/2}{Cn^{1-\theta}\log n}$. ■

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To have witnessed such graphs, for $k = 6, g = 6$, one would have to consider $n \geq 2^{42}$ (!) This explains why it seemed 'counter-intuitive' that large chromatic number and large girth cannot happen simultaneously.

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1. *For every subset H of at most ϵn vertices $\chi(H) \leq 3$.*
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- Proof uses a probabilistic construction.
- Almost every graph (in the random graph model) can be altered mildly to obtain such a G.

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All these proofs heavily rely on probabilistic techniques.

List Colorings of Graphs

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Definition

The **List Chromatic number** *of G (denoted $\chi_l(G)$) := Minimum k for any collection of color lists*

$$\mathcal{L} := \{L_v | v \in V\} \textit{ satisfying } |L_v| \geq k$$

there is a list coloring $\phi_{\mathcal{L}}$.

This should hold irrespective of the actual lists themselves.

If all the lists are identical, then the minimum number k is the definition above is simply $\chi(G)$.

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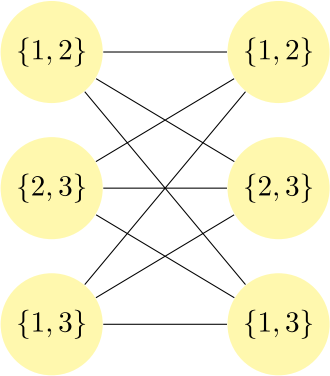

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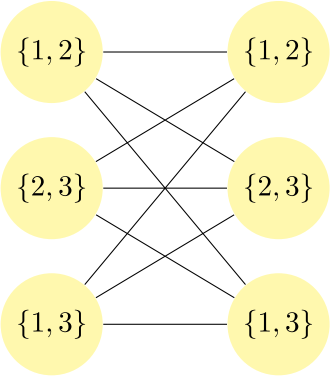

$\chi_l(K_{3,3}) > 2$.

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Theorem

(Johansson,Kim): For $\Delta \gg 0$, $\chi_l(G) \leq O\left(\frac{\Delta}{\log \Delta}\right)$ if G is triangle free (resp. girth at least 5).