Facility Location Problems

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Heron's Problem

- HIGHWAY FACILITY LOCATION

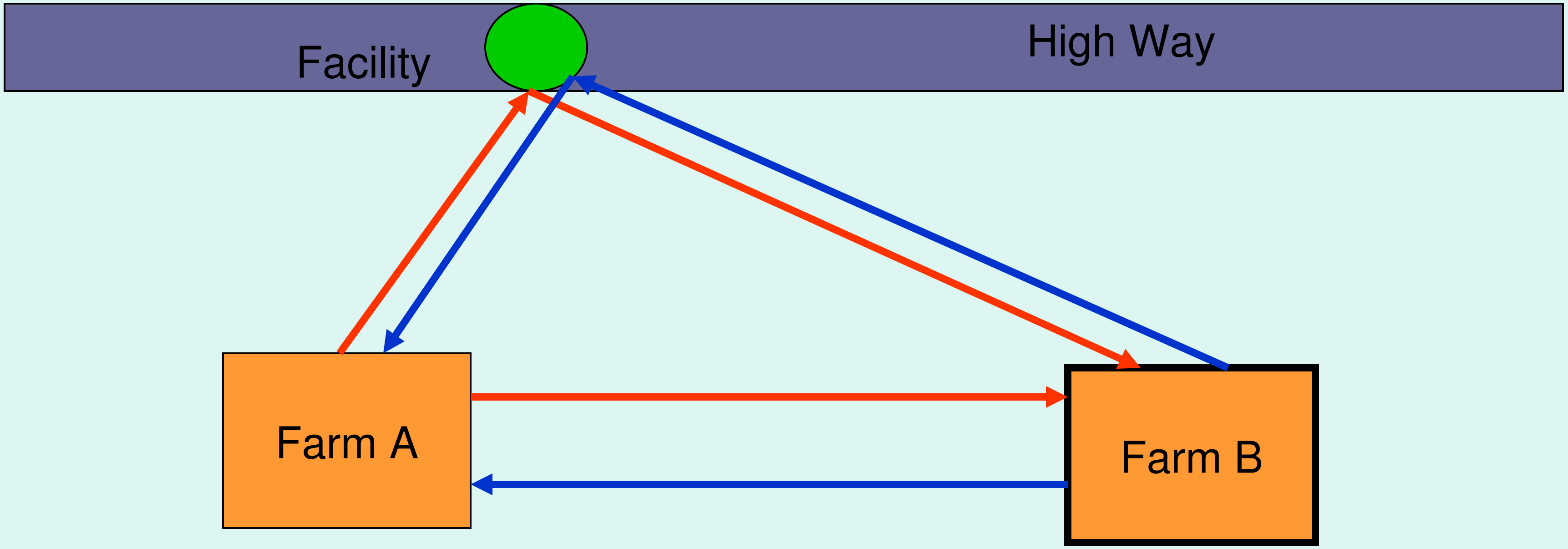
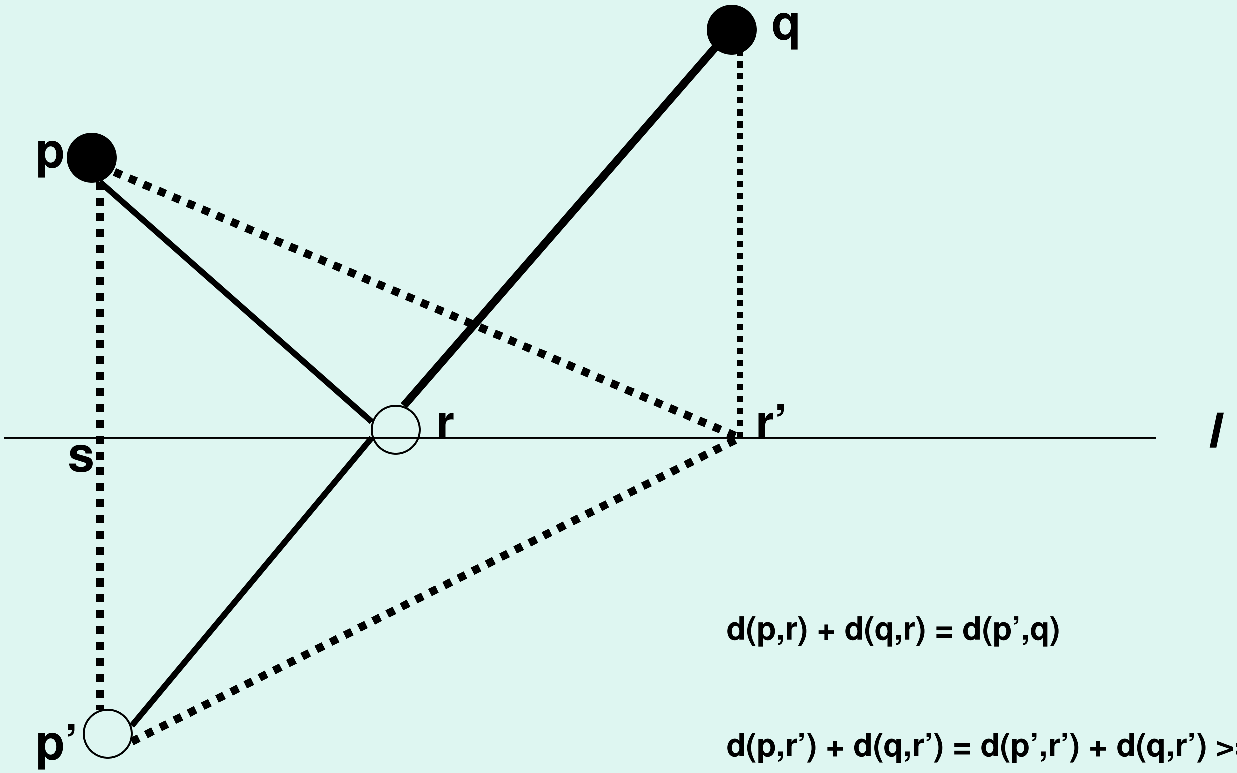


Illustration of the Proof of Heron's Theorem



$d(p,r) + d(q,r) = d(p',q)$

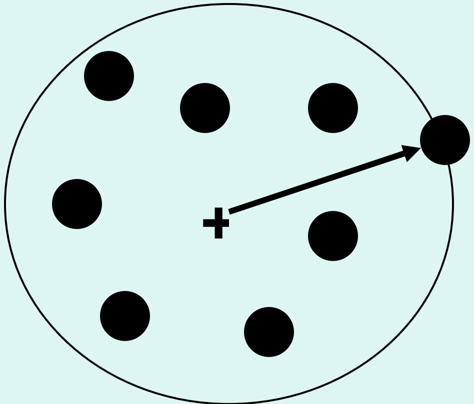
$d(p,r') + d(q,r') = d(p',r') + d(q,r') >= d(p',q)$

MINIMAX FACILITY LOCATION

Given *n* points in the plane representing customers (plants, schools, towns etc..) it is desired to determine the location *X* (another point in the plane) where a facility should be located so as to minimize the distance from *X* to its furthest customer.

The problem has an elegant and succinct geometrical interpretation: "Find the smallest circle that encloses a given set of *n* points."

The center of the circle is Precisely the location of *X*



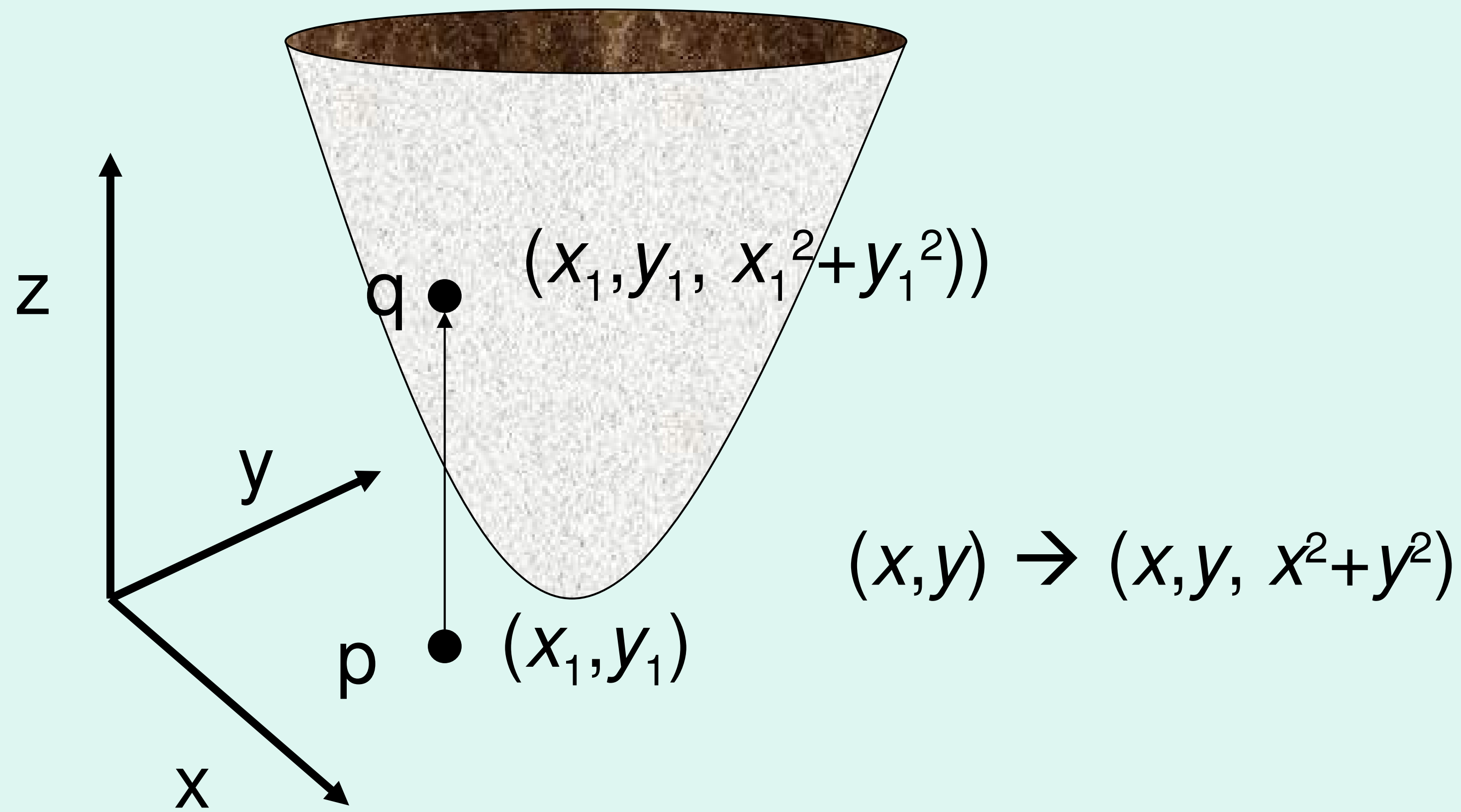
The smallest circle enclosing a set of points

The minimum enclosing circle can be computed from the ***Furthest Neighbor Voronoi Diagram.*** [Shamos]

The correctness of the above algorithm was established by [Bhattacharya & Toussaint]. Their argument is based on the following property of the minimum enclosing circle :

The minimum enclosing circle of a set ***S*** of *n* points is either determined by the diameter of the point set ***S*** or by three points on the convex hull of ***S***, where those three points form an acute angled triangle.

Furthest Neighbor Voronoi Diagram can be computed from the projection of the upper hull of the point set lifted on the paraboloid $z = x^2 + y^2$.



But the computation of ***Furthest Neighbor Voronoi Diagram*** is lower bounded by the computation of the convex hull of *n* points i.e., Ω (*nlogn*).

But this lower bound doesn't hold for the minimum enclosing circle.

In fact we can compute the minimum enclosing circle of a set of *n* points in $\theta(n)$ time by solving three variable **Convex Program**. [Meggido]

The result holds for any fixed dimensional point set.

Here (x_1,y_1), (x_2, y_2), …., (x_n,y_n) are the set of points for which we have to compute the minimum enclosing circle.

Let (x,y) be the center of the minimum enclosing circle and let r be its radius. Then we have the minimum enclosing circle problem formulated as the following optimization problem:

Minimize r^2

Subject to: $(x-x_i)^2 + (y-y_i)^2 \le r^2 \quad i = 1, 2, \ldots, n$

If we substitute $z = x^2 + y^2 - r^2$ then we have the following convex programming formulation of the problem:

Minimize $x^2 + y^2 - z$

Subject to: $-2xx_i -2yy_i + z + (x_i^2 + y_i^2) \le 0 \quad i = 1, 2, \ldots, n$

Thus we have to minimize a quadratic function over a set of linear constraints involving 3 variables.

Fermat-Weber Problem

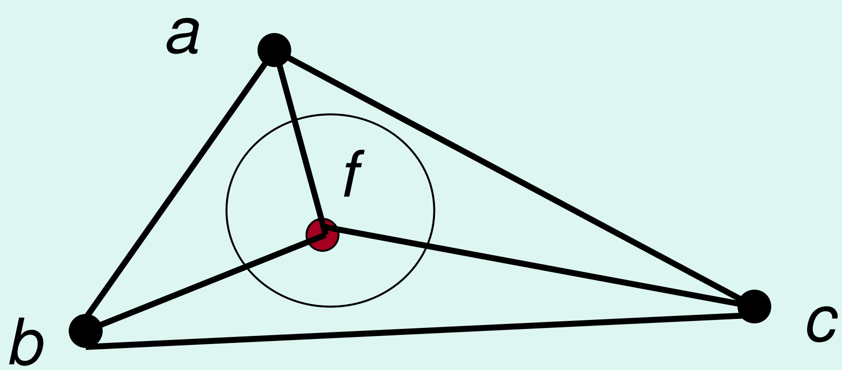
Problem Definition:

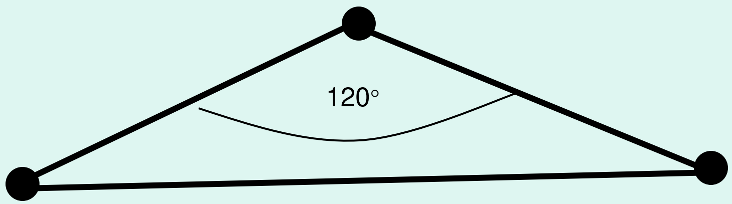
For a given set of *m* points x_1, x_2, …, x_n with each $x_i \in R^d$ find a point y from where the sum of all Euclidean distances to the x_i's is the minimum.

This problem is also known as Geometric 1-Median problem :

$$\underset{y \in R^d}{\arg\min} \sum_{i=1}^{n} \|x_i - y\|$$

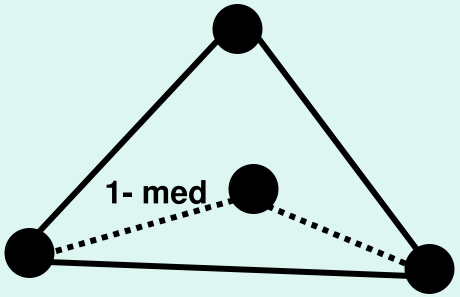
The special case when $n = 3$ and $d = 2$ arises in the construction of Minimal Steiner Trees and was originally posed as problem by Torricelli.

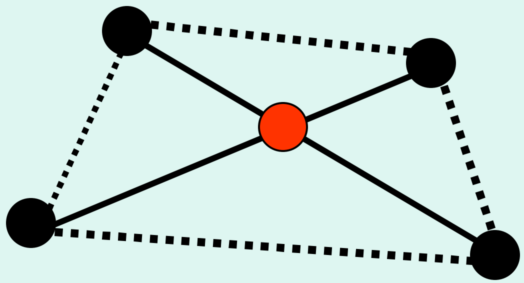




For 3 coplanar points *a*, *b*, *c* if each angle of the triangle *abc* is less tan 120° the Fermat Point *f* is a point inside the triangle which subtends 120° an angle to all three pairs of points. Otherwise if any angle of the triangle *abc* is greater than then Fermat Point is the point making that angle.

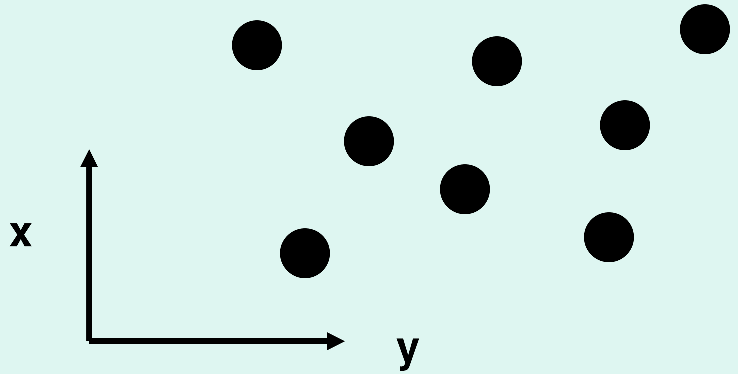
For 4 coplanar points if a point is inside the triangle formed by the other three points then geometric 1-median is that point. Otherwise all 4 points form a convex quadrilateral and the geometric 1-median is the intersection point of both the diagonals and also known as Radon Point.





If we restrict $d = 1$, i.e., for the 1-dimensional case then the geometric 1-median coincides with the median.

Again if we consider the problem in L_1 metric with $d = 2$, i.e., in 2-dimension we can combine the optimal solutions of two 1 dimensional problems to obtain the optimal solution of the original problem.



The method extends for any value of d in L_1 metric ,i.e., we can decompose the problem in d one dimensional problems and combine their optimal solution to obtain the optimal solution of the original problem.

In L_2 metric Weiszfeld's algorithm iteratively computes the geometric 1-median problem :

$$y_{i+1} = \frac{\sum_{j=1}^{n} \frac{x_j}{\| x_j - y_i \|}}{\sum_{j=1}^{n} \frac{1}{\| x_j - y_i \|}}$$

K-Median

Given a set ***S*** of ***n*** points in the plane we have to locate a set ***F*** of ***k*** facilities that partitions the set S in ***k*** partitions $N(f_i)$ i =1,2, …, ***k*** so that for any site s_j $\in$ $N(f_i)$ [neighborhood of f_i] being served by the facility f_i the following sum is minimized:

$$\sum_{i=1}^{k} \sum_{s_j \in N(f_i)} dist(f_i, s_j)$$

where

$$S = \bigcup_{i=1}^{k} N(f_i)$$

and

$$N(f_i) \cap N(f_j) = \phi, 1 \le i < j \le k$$

In ***k***-median problem the elements for the set ***F*** can be any ***k*** sites in R^d in the unrestricted case. Sometimes we restrict the ***k***-median problem such that $F \subseteq S$, i.e., input point sites. Both of these problems are NP-Hard. [Meggido & Supowit]

So our goal should be the design of efficient approximation algorithms for these problems. Here we will restrict our attention to the restricted version of the problem.

Restricted version of our facility location can still be classified into two subcategories:

(a) Uncapacitated ***k***-median problem

(b) Capacitated ***k***-median problem

A common assumption used when dealing with location problems is that facilities are ***uncapacitated*** and can thus service any number of demand destinations. But the assumption may be unrealistic in many applications and thus in ***capacitated*** facility location an upper bound is provided on the number of demand destinations serviced by each facility.

Now we can formulate these problems using ILP as follows:

Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} dist(s_i, s_j) x_{ij}$

Subject to:

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2,\ldots,n$$

$$\sum_{j=1}^{n} x_{ij} \leq C_i \qquad i = 1,2,\ldots,n$$

$$\sum_{i=1}^{n} y_i = k$$

$$y_i - x_{ij} \geq 0 \qquad i = 1,2,\ldots,n \qquad j = 1,2,\ldots,n$$

$$y_i \in \{0,1\}$$

$$x_{ij} \in \{0,1\} \qquad i = 1,2,\ldots,n \qquad j = 1,2,\ldots,n$$

The indicator variable y_i denotes if the site s_i is selected as the facility or not.

The indicator variable x_{ij} denotes if the facility at the site s_i serves the customer at site s_j.

Since ILP is NP-Complete the usual technique is to relax the integrality constraint , i.e., replace the above two constraints by:

$$0 \le y_i \le 1 \quad i = 1,2,\ldots,n$$

$$0 \le x_{ij} \le 1 \qquad i = 1,2,\ldots,n \qquad j = 1,2,\ldots,n$$

Then we solve the LP-relaxation of the ILP. After that the fractional solution is cleverly rounded maintaining the feasibility and some bound is established w.r.t $Z^*_{LP} \leq Z^*_{ILP}$

For rectilinear 1-median problem $\Omega(n\log n)$ lower bound was established by [Bajaj].

As far as upper bounds are concerned there are $O(n^{1.5}\log^4 n)$ time algorithm for arbitrary weighted points and $O(n\log^4 n)$ time algorithm for equally weighted point set was exhibited by [ElGindy & Keil]

Thus even in L_1 metric there are big gaps between these upper and lower bounds.

All these pose several new directions of future research.

Line Facility

In many practical applications we may be interested in locating a line facility rather than locating a point facility.

For example we may have to construct a road in some residential area so that it is convenient to most of the residents. Then this road design is the same as locating a line facility among a set of sites for residents.

There can be many variations of these problems in various metric. The L_2 approximation of this problem, i.e., where we have to minimize the sum of the squares of the perpendicular distances is the well known Regression Line Problem.

Given a set of data points (x_1, y_1), (x_2, y_2), …., (x_n, y_n) we have to find the equation of the straight line $y = ax + b$ that minimizes the L_2 error, i.e.,

Minimize $$\sum_{i=1}^{n} [y_i - (ax_i + b)]^2 = E$$

Here the variables are in fact ***a*** and *b*. So we have two unknowns and we require at least two equations to solve. They are obtained from the minima criterion after taking partial derivatives w.r.t. ***a*** and ***b*** respectively:

$$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0$$

The result can be extended for any dimensional data. For example in ***d***-dimensions we have to find the Regression Hyper-plane $a_1x_1 + a_2x_2 + \ldots + a_dx_d = b$. Thus we have to

Minimize $$\sum_{i=1}^{n} [b - \sum_{j=1}^{d} a_j x_j^i]^2 = E$$

Thus we have ***d*** - unknowns $a_1, a_2, \ldots, a_d$. They are solved from the following system of equations:

$$\frac{\partial E}{\partial a_1}=0, \frac{\partial E}{\partial a_2}=0, \ldots, \frac{\partial E}{\partial a_d}=0$$

For L_1 approximation of the problem we have to

Minimize $$\sum_{i=1}^{n} | b - \sum_{j=1}^{d} a_j x_j^i | = E$$

This absolute value function is not a linear function. But we can easily convert it to a standard LP problem as follows [Chvatal]:

Minimize $$\sum_{i=1}^{n} e_i$$

Subject to: $$b - \sum_{j=1}^{d} a_j x_j^i \le e_i \qquad i = 1, 2, \ldots, n$$

$$-b + \sum_{j=1}^{d} a_j x_j^i \le e_i \qquad i = 1, 2, \ldots, n$$

For L_∞ approximation of the problem we have to

Minimize $$\max_i |b - \sum_{j=1}^{d} a_j x_j^i|$$

This can be directly formulated into LP as follows:

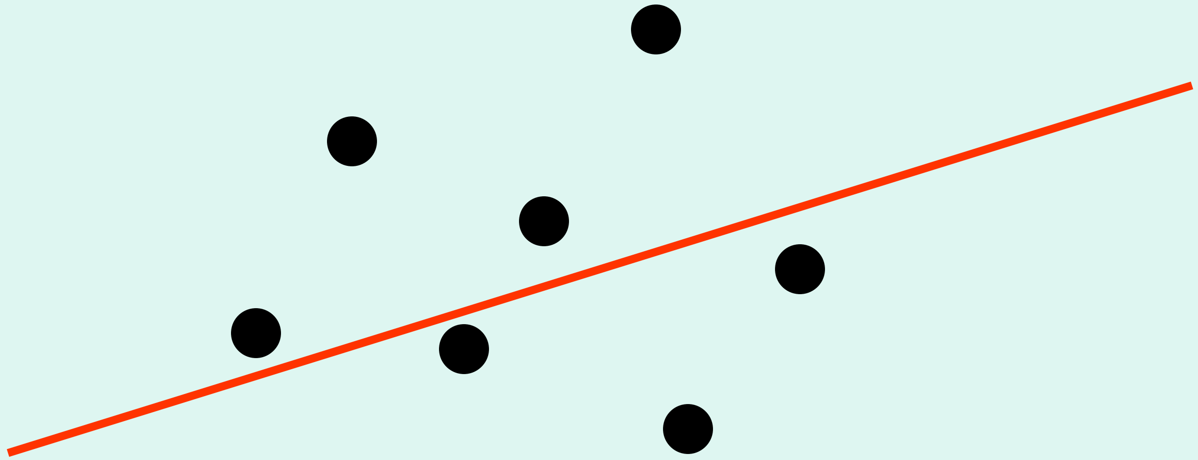
Minimize $$z$$

Subject to: $$z + \sum_{j=1}^{d} a_j x_j^i \geq b \qquad i = 1, 2, \ldots, n$$

$$z - \sum_{j=1}^{d} a_j x_j^i \geq -b \qquad i = 1, 2, \ldots, n$$

The problem of L_1 and L_∞ approximation problem was first posed By Fourier. Subsequently many good algorithms for computing L_1 and L_∞ approximation were proposed by [Bloomfield & Steiger].

Similar type of problems for line facility concerns:



i) Locating a line facility that minimizes the maximum distance from a set of sites.

ii) Locating a line facility that minimizes the sum of the distances from a set of sites.

There are several possible extensions of these problems.

THANK YOU