Duality Transformation and its Applications to Computational Geometry

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Outline

Introduction

- In the Cartesian plane, a point has two parameters (x- and y-coordinates) and a (non-vertical) line also has two parameters (slope and y-intercept). We can thus map a set of points to a set of lines, and vice versa, in an one-to-one manner.

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- The concept of duality is a powerful tool for the description, analysis, and construction of algorithms.
- In this lecture we explore how geometric duality can be used to design efficient algorithms for a number of important problems in computational geometry.

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- Each such mapping has its advantages and disadvantages in particular contexts.

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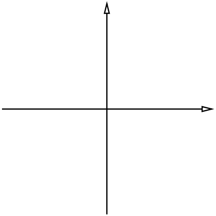
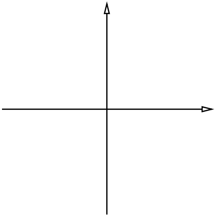


primal



dual

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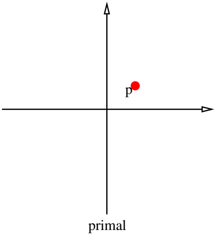
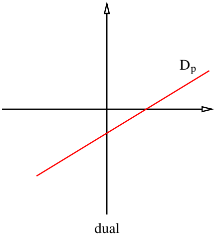
A point $p(a, b)$ is transformed to the line $D_p(y = ax - b)$.





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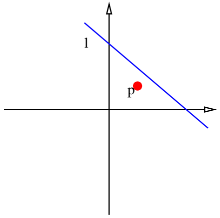

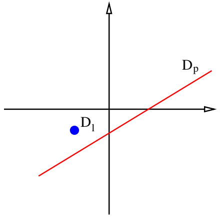

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A point $p(a, b)$ is transformed to the line $D_p(y = ax - b)$.

Definition

A line $l(y = cx + d)$ is transformed to the point $D_l(c, -d)$.

Properties

Lemma

D is its own inverse, that is, $DD_p = p$ and $DD_l = l$.

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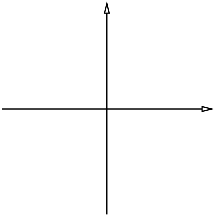
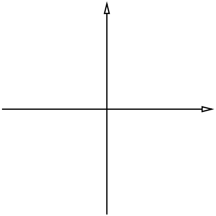
However this is not a problem in general. Because we can always rotate the problem space slightly so that none is vertical.

Properties

Incidence preserving



primal



dual

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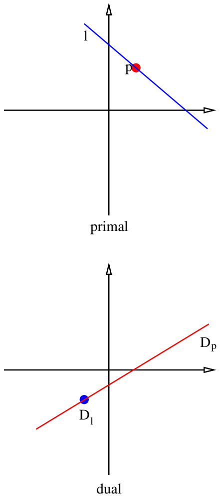
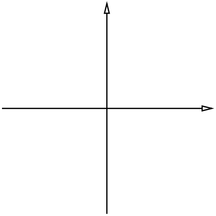


Lemma

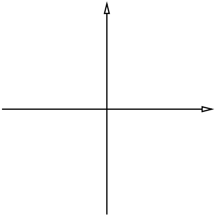
A point $p(a, b)$ is incident to the line $l(y = cx + d)$ in the primal plane iff point $D_l(c, -d)$ is incident to the line $D_p(y = ax - b)$ in the dual plane.

Properties

Order preserving



primal



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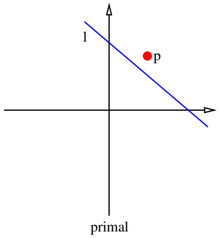
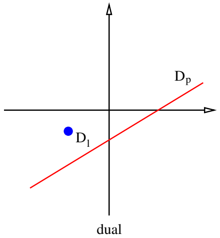


Lemma

A point $p(a, b)$ is above (below) the line $l(y = cx + d)$ in the primal plane iff line $D_p(y = ax - b)$ is below (above) the point $D_l(c, -d)$ in the dual plane.



Alternative Definition

- The duality transformation we have described so far is often called m-c duality.

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- An alternative definition, called polar duality, is also used.

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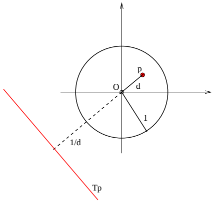
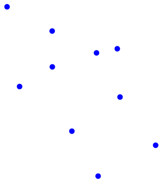


Outline

Convex Hull

Let $\mathcal{P}$ be a set of points in the plane.



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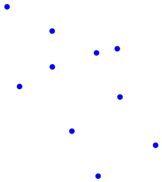
Definition

Convex hull of $\mathcal{P}$, denoted by $CH(\mathcal{P})$, is the smallest convex set containing $\mathcal{P}$.



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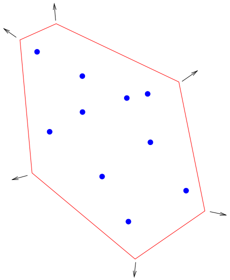

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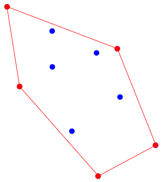

Convex Hull Problem

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To compute the convex hull of a point set is a well known and fundamental problem in computational geometry.

A Naive Algorithm

`input: a set S of n points in the plane`

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input: a set S of n points in the plane
for each triple (u,v,w) of S{

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input: a set S of n points in the plane
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input: a set S of n points in the plane
for each triple (u,v,w) of S{
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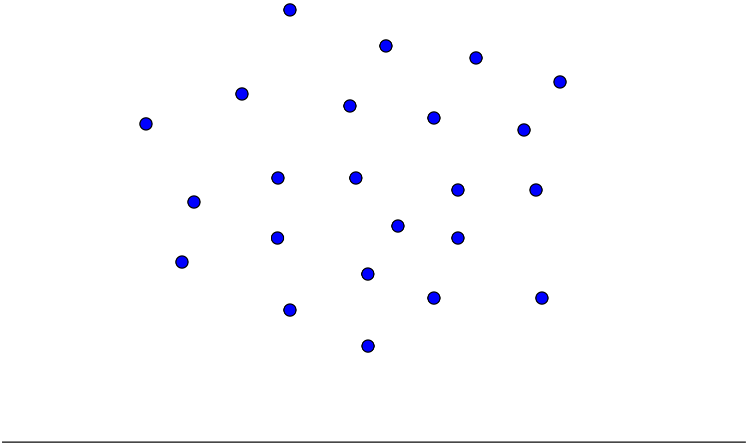
}
Order the remaining points of S and
  output the ordered list
```

Example



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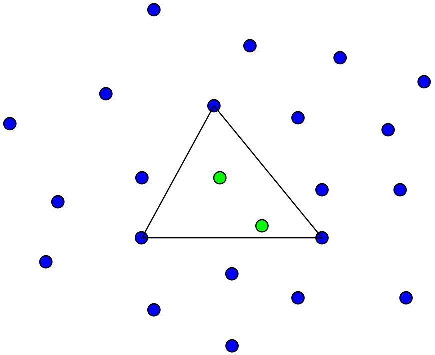

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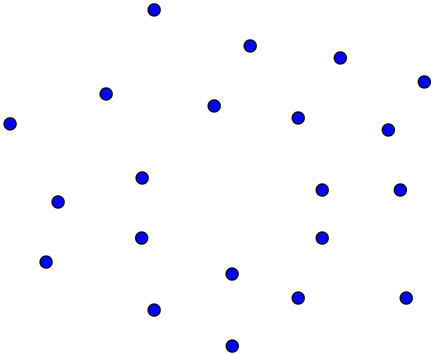

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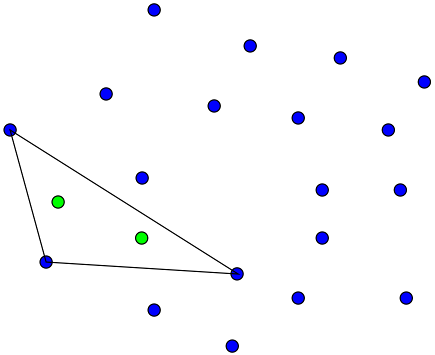

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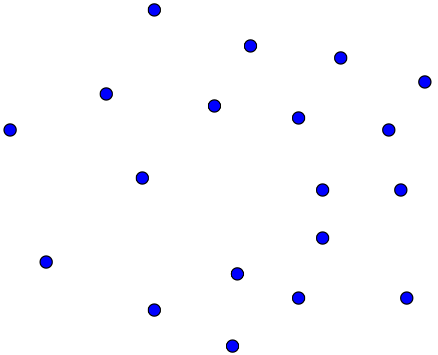

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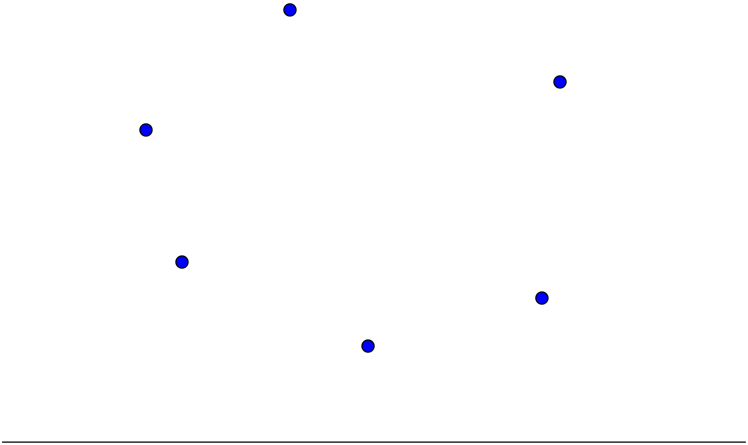

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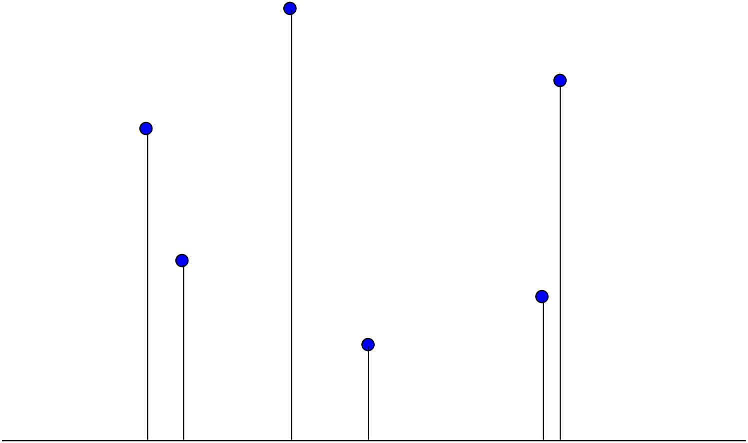

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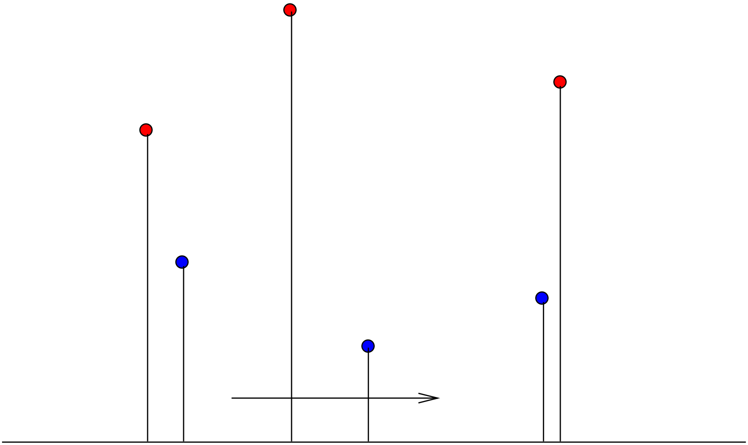

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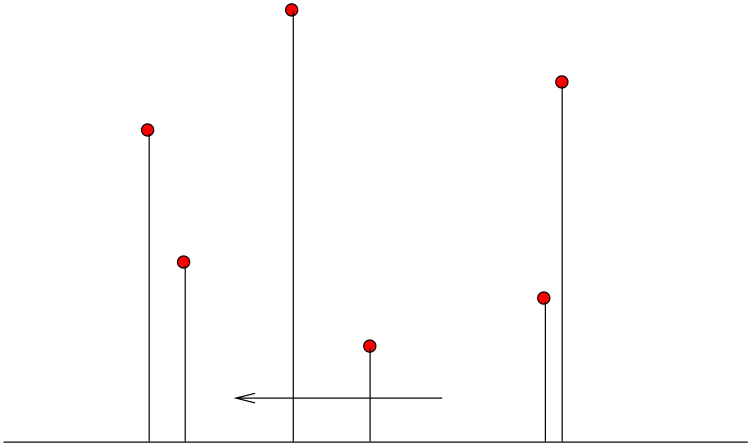

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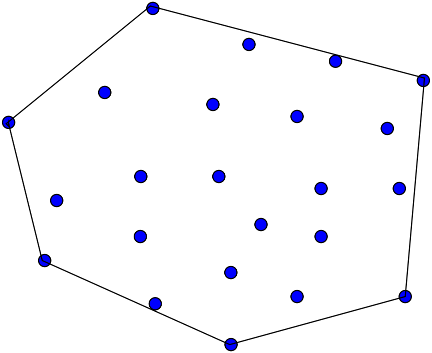

Complexity

There are $O(n^3)$ triangles and it takes $O(n)$ time for each triangle.
So processing time for all triangles is $O(n^4)$.
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Result

The naive algorithm takes $O(n^4)$ time and $O(n)$ space to compute the convex hull of a set of n points.

Optimal Algorithms

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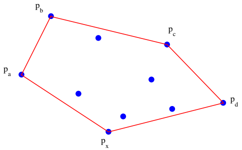
- A number of optimal algorithms have been devised for the convex hull problem.

Optimal Algorithms

- Grahams scan, time complexity $O(nlogn)$.
  (Graham, R.L., 1972)
- Divide and conquer algorithm, time complexity $O(nlogn)$.
  (Preparata, F. P. and Hong, S. J., 1977)
- Jarvis's march or gift wrapping algorithm, time complexity $O(nh)$ where h number of vertices of the convex hull.
  (Jarvis, R. A., 1973)
- Most efficient algorithm to date is based on the idea of Jarvis's march, time complexity $O(nlogh)$.
  T. M. Chan (1996)

Definitions

Let $\mathcal{P}$ be the given set of n points in the plane. Let $p_a \in \mathcal{P}$ be the point having smallest x-coordinate and $p_d \in \mathcal{P}$ be the point with largest x-coordinate. Obviously, both p_a and p_d belongs to $CH(\mathcal{P})$.



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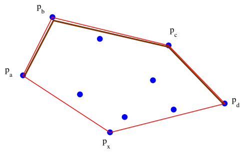
Definition

The c-wise polygonal chain $p_a, \ldots, p_d$ along the hull is called the upper hull.



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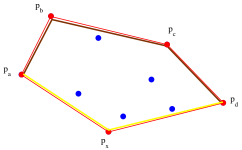

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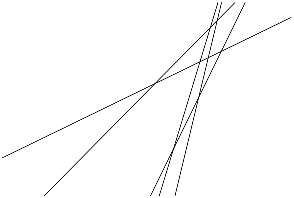
Definition

The cc-wise polygonal chain $p_a, \ldots, p_d$ along the hull is called the lower hull.

Definitions

Let $\mathcal{L}$ be a set of lines in the plane.



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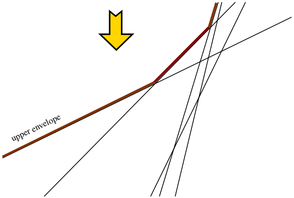
The upper envelope is a polygoal chain E_u such that no line $l \in \mathcal{L}$ is above E_u.



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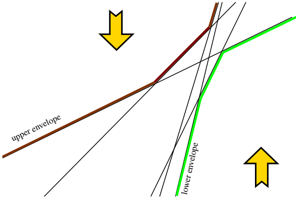
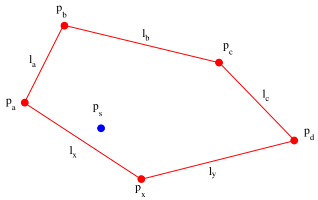


Connection Between Hull and Envelope



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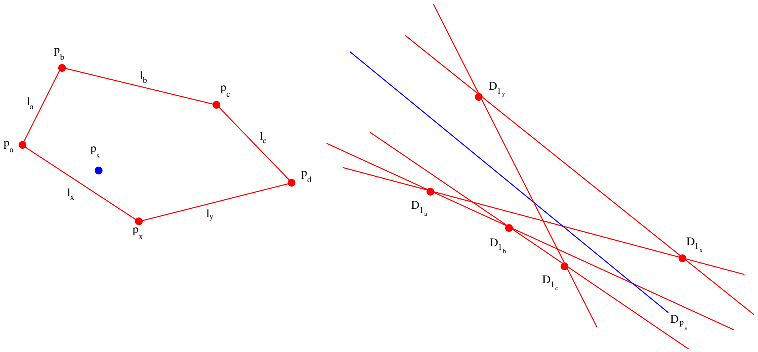

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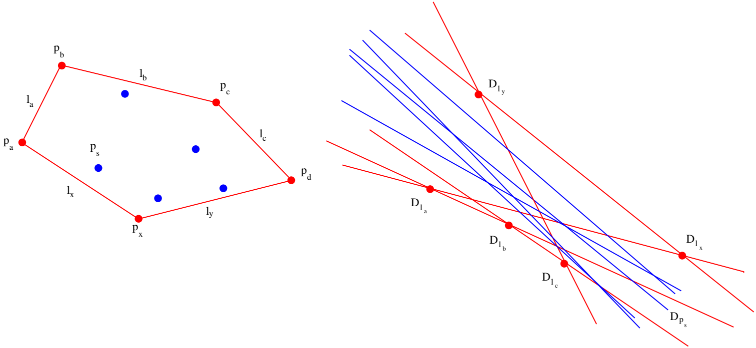

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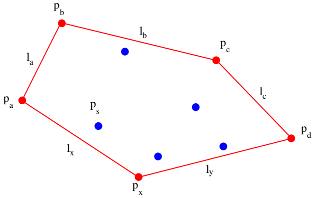

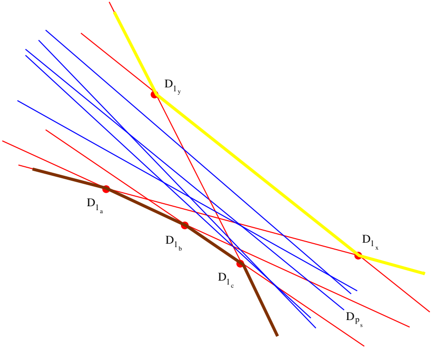

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Conclusion

Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.

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Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.

Thus the problem of computing convex hull of a point set in the primal plane reduces to the problem of computing upper and lower envelopes of the line set in the dual plane.

Algorithm

```
Input: I = (L1, L2, ..., Ln) is the list of dual lines
       in the increasing order of slopes.
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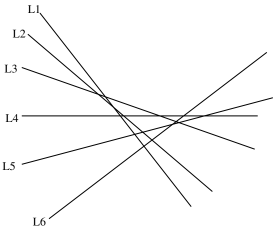
  insert the line Li at the tail of the list O;
}
```

Example



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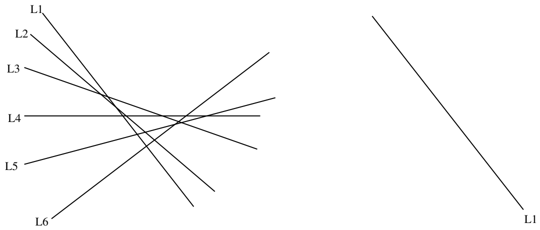
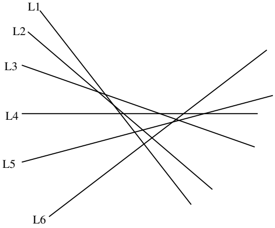
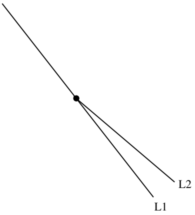


Example





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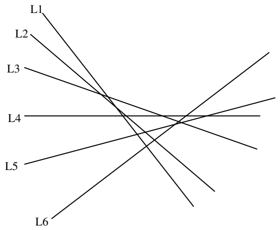

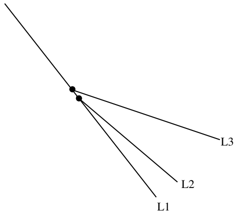

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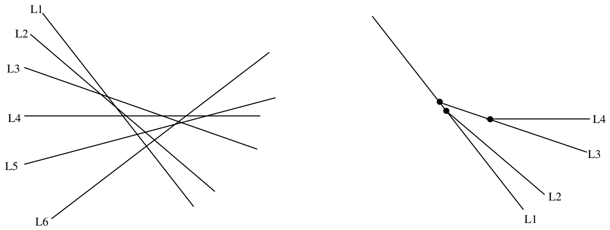

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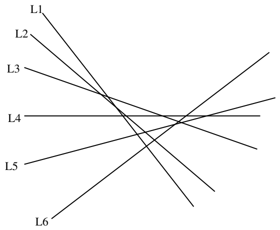

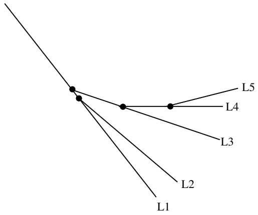

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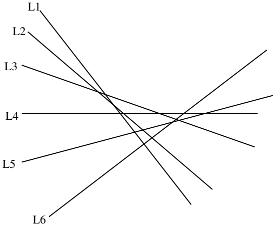

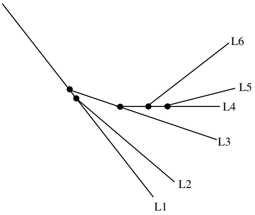

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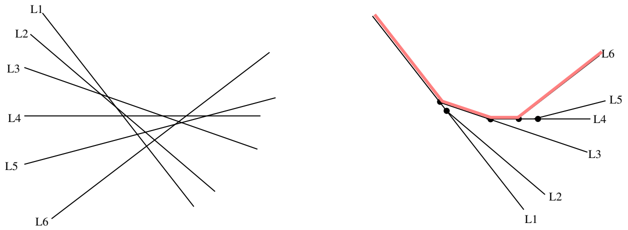

Result

Lemma

After sorting n lines by their slopes in $O(nlogn)$ time, the upper envelope can be obtained in $O(n)$ time.

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Proof.

It may check more than one line segment when inserting a new line, but those ones checked are all removed except the last one. □

Result

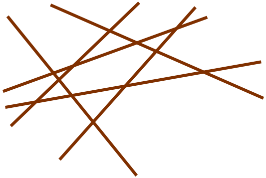
Result

Given a set $\mathcal{P}$ of n points in the plane, $CH(\mathcal{P})$ can be computed in $O(n \log n)$ time using n space.

Outline

Definition

- Let $\mathcal{L}$ be a set of n lines in the plane. The embedding of $\mathcal{L}$ in the plane induces a planner subdivision that consists of vertices, edges, and faces where some of the edges and faces are unbounded. This subdivision is referred to as arrangement induced by $\mathcal{L}$, and is denoted by $A(\mathcal{L})$.



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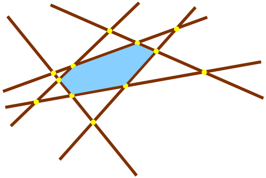

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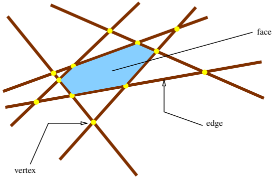

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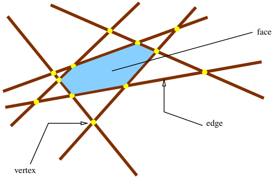
- An arrangement is called simple if no three lines passes through the same point and no two lines are parallel.



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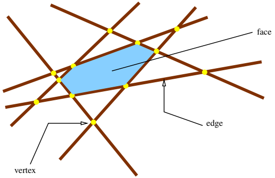
- An arrangement is called simple if no three lines passes through the same point and no two lines are parallel.
- The (combinatorial) complexity of an arrangement is the total number of vertices, edges, and faces. Observe that worst case complexity occurs when an arrangement is simple.



Result

Theorem

Let $\mathcal{L}$ be the set of n lines in the plane, and let $A(\mathcal{L})$ be the arrangement induced by $\mathcal{L}$.

(i) *The number of vertices of $A(\mathcal{L})$ is at most $n(n-1)/2$.*

(ii) *The number of edges of $A(\mathcal{L})$ is at most n^2.*

(iii) *The number of faces of $A(\mathcal{L})$ is at most $n^2/2 + n/2 + 1$.*

Equality holds in these three statements iff $A(\mathcal{L})$ is simple.

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Can be proved easily by using Euler's formula:
For any connected planner embedded graph with m_v nodes, m_e arcs, and m_f faces the following relation holds

$$m_v - m_e + m_f = 2.$$

Computation of Arrangement

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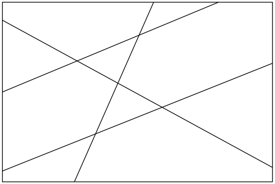
- Algorithms for a surprising number of problems are based on constructing and analyzing the arrangement of a specific set of lines.
- A variety of data structures have been proposed for this purpose.

Computation of Arrangement

- Algorithms for constructing arrangements are usually incremental. Beginning with an arrangement of one or two lines, subsequent lines are inserted into the arrangement one at a time, thereby building larger and larger arrangements.



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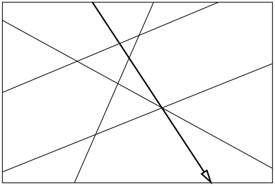

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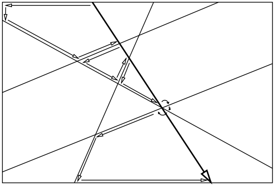

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- We can thus scan through each edge of every cell encountered on our insertion walk in linear time.
- The total time to insert all n lines in constructing the full arrangement is $O(n^2)$.

Levels

- We consider an alternative concept, called levels, for structuring an arrangement of lines.

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- It is simple both from understanding and implementations point of view.

Definition

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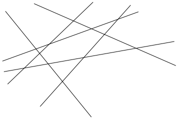
Let $\mathcal{L}$ be a set on n lines in the plane inducing an arrangement $A(\mathcal{L})$. A point π in the plane is at level θ ($0 \leq \theta \leq n$) if there are exactly θ lines in $\mathcal{L}$ that lie strictly below π. The θ-level of $A(\mathcal{L})$ is the closure of a set of points on the lines of $\mathcal{L}$ whose levels are exactly θ in $A(\mathcal{L})$, and is denoted as λ_θ.

definition



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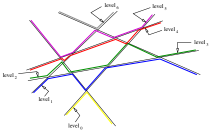
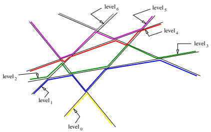


definition

- Clearly, the edges of λ_θ form a monotone polychain from $x = -\infty$ to $x = \infty$. Each vertex of the arrangement $A(\mathcal{L})$ appears in two consecutive levels, and each edge of $A(\mathcal{L})$ appears in exactly one level.



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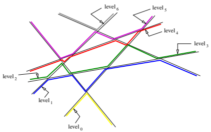

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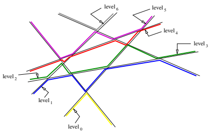
- We can thus store each level simply as an array of segments.
- Observe that the upper and the lower envelops mentioned earlier, are simply the n-th and 0-th levels respectively.



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- Here we consider an alternative method using plane sweep paradigm.
- The method was first introduced by Bentley and Ottmann (1979) in the context of solving the problem of line segment intersections.

Plane Sweep Method

Basic method consists of:

- A vertical line l, called the sweep line, sweeps over the plane from $x = -\infty$ to $x = \infty$.

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- The algorithm performs some computational steps when the sweep line reaches event points.

Data structure

- In our case, events are the points of intersection of the lines. Apart from storing the events, we also need to insert new events and extract the event nearest to the sweep line on its right. Clearly, a queue is a suitable data structure for this.

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- At every instant, the sweep line intersects each element of $\mathcal{L}$. Order the lines from bottom to top according to their intersection points with the sweep line. Data structure we use for maintaining the sweep line status are arrays storing the levels. At an instant, portion of the line at the i-th position, $0 \leq i \leq n$, is part of the i-th level.

Processing

- Let the next event be the intersection point of the lines currently at i-th and $(i + 1)$-th positions respectively. Processing steps to be performed at this event point are as follows.

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- If the line at the $(i+1)$-th position after the event point intersect the line at the $(i+2)$-th position on the right of the sweep line, then insert the intersection point as a future event point. Similarly, if the line at the i-th position after the event point intersect the line at the $(i-1)$-th position on the right of the sweep line, then insert the intersection point as a future event point.

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- Check each pair of lines from bottom to top if they insert on the right of the sweep line. If yes, insert these intersection points in the queue as an event point.

Algorithm

`Input: A set L of n lines in the plane`

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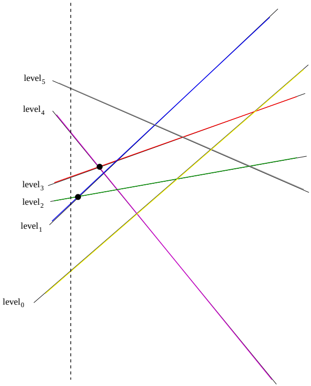

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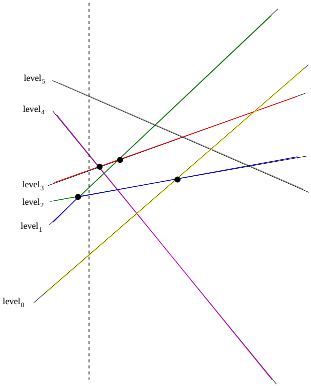

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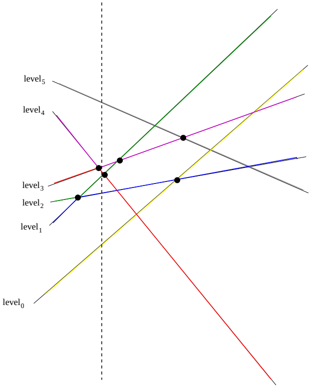

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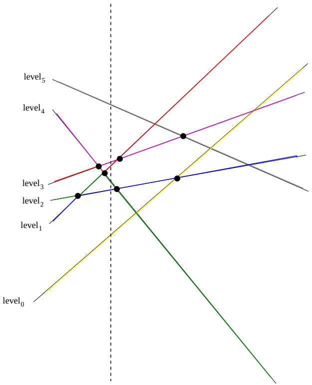

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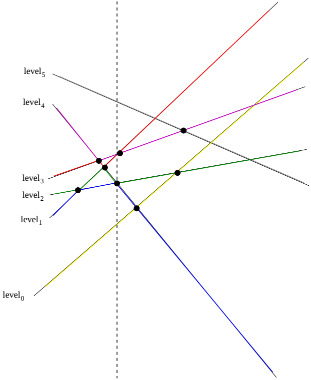

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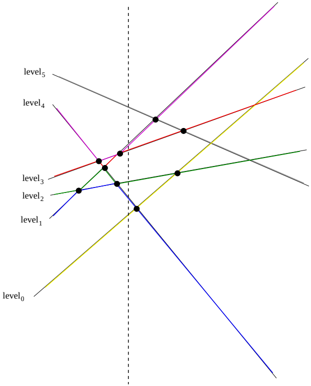

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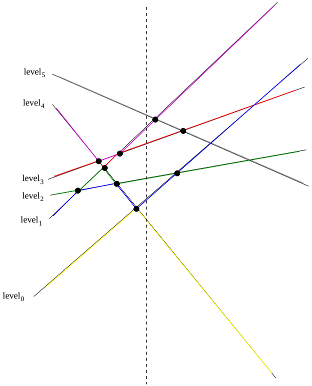

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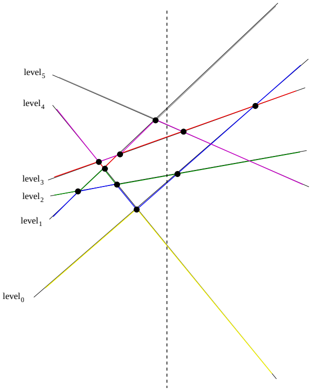

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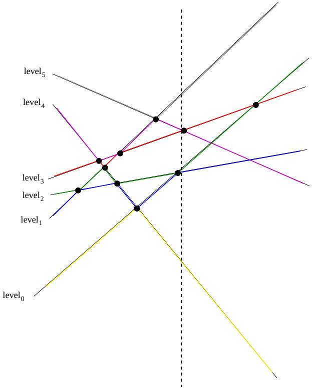

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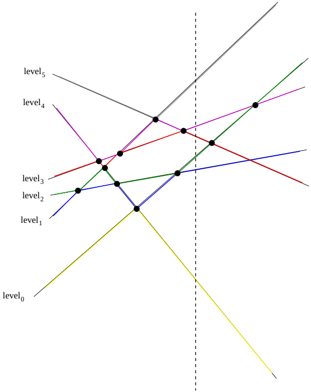

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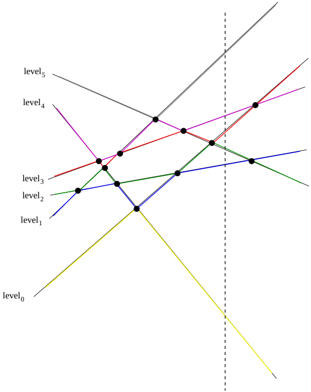

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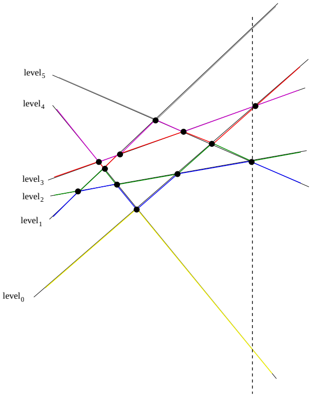

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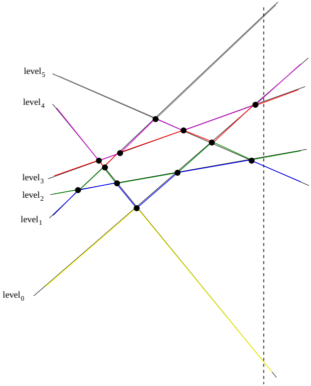

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- Space complexity is $O(n^2)$.

Result

Theorem

Using plane sweep, levels of an arrangement of n lines can be computed in $O(n^2 \log n)$ time using $O(n^2)$ space.

Outline

Smallest Area Triangle Problem

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Let $\mathcal{P}$ be a set of n points in the plane. The problem is to determine which of the $\binom{n}{3}$ triangles with vertices in $\mathcal{P}$ has the smallest area.

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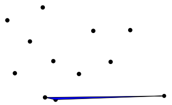
The solution of the above problem allows us to solve the following problem also.

Problem

Let $\mathcal{P}$ be a set of n points in the plane. The problem is to determine whether three points in $\mathcal{P}$ are collinear.

Smallest Area Triangle Problem

- The difficulty of the problem arises from the fact that the vertices of the smallest triangle can be arbitrarily apart (i.e., absence of locality).



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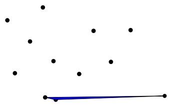

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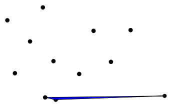
- The best known algorithm, without using duality, for this problem has time and space complexities $O(n^2 \log n)$ and $O(n)$ respectively. (Edelsbrunner and Welzl, 1982).
- Using duality, it is possible to improve upon the complexity.



Assumption

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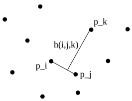
- To avoid this we assume that no two points in $\mathcal{P}$ have same x-coordinates. This may possibly require rotating the axes by a small angle which can be determined in $O(n \log n)$ time.

Sketch of the solution

- Let $h(i, j, k)$ be the perpendicular distance from the point p_k to the segment $p_i p_j$ and let the line through p_k that is parallel to $p_i p_j$ is $l(i, j, k)$.



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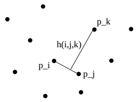

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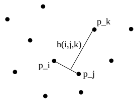

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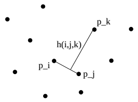
- However, when taken to dual plane, this leads to efficient algorithm.



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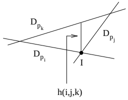
- In the dual plane, the edge $p_i p_j$ becomes the intersection point I of D_{p_i} and D_{p_j}.



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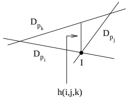

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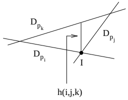
- Expression for the vertical distance $h(i,j,k)$ between I and D_{p_k} is



$$h(i,j,k) = y(p_k) + \frac{x(p_k)[y(p_j) - y(p_i)] + x(p_j)y(p_i) - x(p_i)y(p_j)}{x(p_i) - x(p_j)}$$

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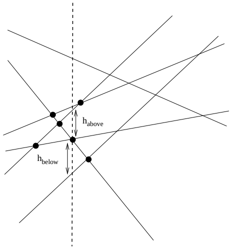

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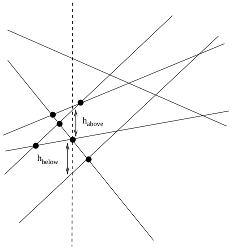
- Let the minimum distance occurs for the line D_{p_k}. Compute the minimum area of the triangle with $p_i p_j$ as base.



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- Hence space complexity of the algorithm is $O(n)$.
- Time complexity of the algorithm is, clearly, $O(n^2 \log n)$.
- The $\log n$ factor in the time complexity can be avoided by using topological line sweep.
  (Edelsbrunner, H. and Guibas, L. J., 1989)

Outline

Problem

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Given a set $\mathcal{P}$ of n points in the plane and a query line l, compute the nearest neighbor (in the perpendicular distance sense) of the query line l.

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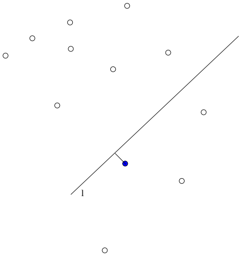

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- Here we are allowed to preprocess the point set so that each query can be answered efficiently.

Strategy

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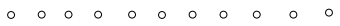
- Since our definition of duality does not allow vertical line, we need to have separate algorithm for handling vertical query lines.

Nearest Neighbor Query Vertical Line

- Sort the points of the given set $\mathcal{P}$ on their x-coordinates. This can be done in $O(n \log n)$ time using $O(n)$ space.



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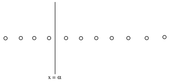

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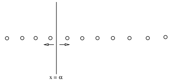

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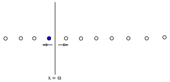

Farthest Neighbor Vertical Query Line

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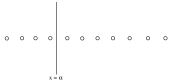

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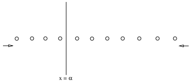

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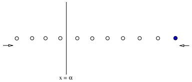

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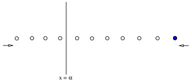

Result

Lemma

With $O(n \log n)$ preprocessing time using $O(n)$ space, nearest and farthest neighbors of a query vertical line can be found in $O(\log n)$ time.

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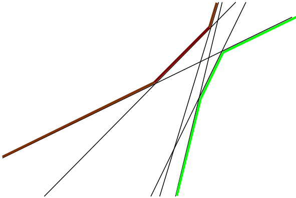
- As the preprocessing step, compute the upper envelope and the lower envelope of the set of lines dual to the given set of points $\mathcal{P}$. This can be done in in $O(n \log n)$ time using $O(n)$ space as mentioned previously.

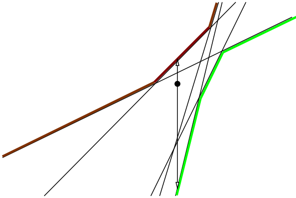
Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.



Farthest Neighbor of a Non-Vertical Query Line

- Let E_u and E_l are the arrays storing the upper and the lower envelops respectively.
- Given a query line l, shoot a vertical ray from the point D_l in upward and downward direction and find the intersection points with the upper and the lower envelope respectively.



Farthest Neighbor of a Non-Vertical Query Line

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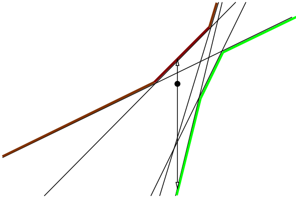
- This can be done in $O(\log n)$ time by using two binary searches on the arrays E_u and E_l holding the envelopes.



Result

Lemma

With $O(n \log n)$ preprocessing time using $O(n)$ space, farthest neighbors of a query non-vertical line can be found in $O(\log n)$ time.

Nearest Neighbor of a Query Non-vertical Line

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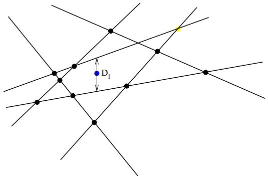
- Let $A(\mathcal{L})$ be the arrangement of lines of the set $\mathcal{L}$.
- Let f be the cell of the arrangement $A(\mathcal{L})$ containing D_l.
- Then one of the points corresponding to the lines just above D_l is the nearest neighbor of l in the primal plane.



Point Location Problem

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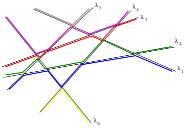
- So with standard data structure, nearest neighbors of a non-vertical query line can be determined in $O(\log n)$ time. The reqired preprocessing time and space is $O(n^2)$.
- Here we describe an algorithm for point location using levels of arrangement.

Point Location Using Level Structure

- First compute the levels of the arrangement $A(\mathcal{L})$ in $O(n^2 \log n)$ time using $O(n^2)$ space.
- Let λ_θ be the linear array containing vertices and edges of level θ, $\theta = 0, 1, \ldots, n$, of the arrangement $A(\mathcal{L})$.



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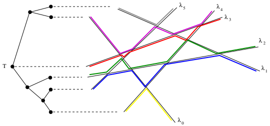
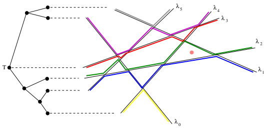
- Let λ_θ be the linear array containing vertices and edges of level θ, $\theta = 0, 1, \ldots, n$, of the arrangement $A(\mathcal{L})$.
- Create a balanced binary search tree T, called the primary structure, whose nodes correspond to the levels θ, $0 \leq \theta \leq n$. Each node of T, representing a level θ, is attached with the corresponding array λ_θ, called the secondary structure. This requires $O(n \log n)$ time and $O(n)$ space.



Point Location Using Level Structure

- Given the query line l, we perform two level binary search on the tree T with the point D_l.



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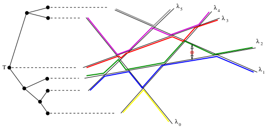

Point Location Using Level Structure

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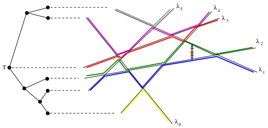
- Time complexity for performing this point location is $O(\log^2 n)$.



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- It may be mentioned that the query time complexity can be reduced to $O(\log n)$, by using a data structuring technique, called fractional cascading.
(Lueker, G. S., 1978)

Thank you!