Computing the maximum clique in visibility graphs

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Overview

1. What is visibility graph?
4. Recognition problem.
5. Characterization problems.
7. Recognition and characterization problems.
8. An algorithm for computing a maximum clique.
9. Open problem
What is visibility graph?

Let $P$ be a polygon with or without holes.

Two points $u$ and $w$ of $P$ are said to be visible if the line segment $uv$ is lies totally inside $P$.

Construct a graph $G$ from $P$ such that every vertex of $P$ is represented as a node in $G$ and two nodes are connected in $G$ iff their corresponding vertices in $P$ are visible from each other.

The graph $G$ is called the visibility graph of $P$.

Minimum dominating set and vertex guard problems

- The minimum dominating set problem in a visibility graph $G$ is to locate a set $S$ of nodes of $G$ of the smallest size such that every node in $G$ is connected by an edge in $G$ to some vertex of $S$.

- The corresponding problem for a polygon $P$ with or without holes is to locate a set $S'$ of vertices of $P$ of the smallest size such that every vertex of $P$ is visible from some vertex of $P$ (Art Gallery problem).

- Lin and Skiena showed that the problems of finding a minimum vertex cover and a maximum dominating set in the visibility graph of a simple polygon are NP-hard.

Maximum hidden set problem

1. The maximum independent set or hidden set problem in a visibility graph $G$ is to locate a set $S$ of nodes of $G$ of the largest size such that no two nodes in $S$ are connected by an edge in $G$.
2. Shermer showed that the maximum hidden set problem is NP-hard.
3. Hidden set has also been studied by other researchers.


Let $G$ be a graph. The problem of determining, if there is some polygon $P$ that has $G$ as its visibility graph, is called the visibility graph recognition problem.

Observe that the boundary of $P$ corresponds to a Hamiltonian cycle of $G$.

The given graph $G$ can have several Hamiltonian cycles.
The given graph $G$ is not the visibility graph of any polygon $P$ if the boundary of $P$ corresponds to the Hamiltonian cycle $(v_1, v_5, v_9, v_3, v_4, v_6, v_8, v_{10}, v_2, v_7)$.

**Problem:** Given an undirected graph $G$ with a Hamiltonian cycle, the problem of recognizing visibility graphs is to test whether there exists a simple polygon such that

1. the Hamiltonian cycle of the graph forms the boundary of a simple polygon,
2. two vertices of the simple polygon are visible if and only if they correspond to adjacent vertices in the graph.


Characterization problem

Characterizing visibility graphs of simple polygons is an open problem.

Ghosh tried to characterize visibility graphs in term of perfect graphs, circle graphs or chordal graphs.

Coullard and Lubiw used clique ordering properties in their attempt to characterize visibility graphs.


Everett and Corneil attempted to characterize visibility graphs using forbidden induced sub-graphs.

Abello and Kumar attempted to characterize visibility graphs using Euclidean shortest paths.

**Open problem:** In-spite of several attempts by many researchers in the last three decades, the problems of recognizing, characterizing and reconstructing visibility graphs of simple polygons are still open.

The maximum clique in $G$ is a complete subgraph of $G$ having the maximum number of vertices.

Vertices $v_1, v_3, v_5, v_6, v_7, v_8, v_9, v_{10},$ and $v_{11}$ have formed a maximum clique in $G$.

The maximum clique in $G$ corresponds to the largest empty convex polygon in $P$ that can be inscribed inside $P$ with the maximum number of vertices of $P$. 
Previous results

- The problem of computing the maximum clique in a visibility graph $G$ of a polygon $P$ with holes is known to be NP-hard.
- It is not clear whether the problem is NP-hard for visibility graphs $G$ of a set disjoint line segments.
- The problem is not known to be NP-hard if the given graph $G$ is the visibility graph of a simple polygon.
- On the other hand, there is no polynomial time algorithm for computing a maximum clique in the visibility graph $G$ of a simple polygon $P$.

Given a set $S$ of $n$ points, the largest empty convex polygon in $S$ can be computed in $O(n^3)$ by the algorithm of Avis and Rappaport.

If the visibility graph $G$ and its corresponding polygon $P$ are given, the maximum clique in $G$ of $P$ can be computed in $O(n^3)$ time by the algorithm of Eidenbenz and C. Stamm.


Can we construct the corresponding simple polygon $P$ of a given visibility graph $G$ and then compute the maximum clique in $G$?

However, there is no polynomial time algorithm known for constructing the corresponding simple polygon $P$ from a given visibility graph $G$.

Therefore, it is not possible to compute the maximum clique in $G$ without having any geometric information of the corresponding polygon $P$. 
With additional information

- Assume that the Hamiltonian cycle $C$ in the visibility graph $G$ corresponding to the boundary of $P$ is given along with $G$.
- With this additional information, the maximum clique in $G$ can be computed in $O(n^2e)$ time by the algorithm of Ghosh et al., where $n$ and $e$ are number of vertices and edges in $G$ respectively.
- Note that there is no polynomial time algorithm known for locating the Hamiltonian cycle $C$ in $G$ that corresponds to the boundary of $P$.


Decomposition of $G$ in $G_1, G_2, \ldots, G_n$

- For each vertex $v_i$ in $G$, let $G_i$ denote the induced subgraph of $G$ formed by $v_i$ and the neighbors of $v_i$ in $G$.
- For example, $G_3$ is the entire visibility graph $G$.
- Observe that if $v_i$ is a vertex of the maximum clique in $G$, then all vertices of the maximum clique in $G$ belong to $G_i$.
- Our algorithm computes the maximum clique in each $G_i$ and then chooses the one which is the largest.
The region of $P$, whose visibility graph is $G_i$, is called the fan $F_i$ with $v_i$ as the fan vertex.

For example, the boundary of $F_{10}$ consists of $v_{10}, v_{11}, v_1, v_2, v_3, v_9$ and $v_{10}$.

Observe that the order of vertices on the boundary of $F_i$ follows the order of vertices in $C$.

If the internal angle $\theta_i$ at $v_i$ in $F_i$ is convex (i.e., $\theta_i \leq 180^\circ$), $F_i$ is called a convex fan.

Can we identify those vertices of $G_i$ that are reflex in $F_i$?
Partition using cross-visibility

- The vertex $v_j$ is not reflex in $F_i$ as $v_i$ is used to block the visibility between $v_{j-1}$ and $v_{j+1}$.
- There is no cross-visibility across $(v_i, v_j)$ as well as across $(v_i, v_p)$.

**Lemma:** The graph $G_i$ can be partitioned recursively into subgraphs using cross-visibility such that the maximum clique in $G_i$ belongs to one of the subgraphs.

Identification of reflex vertices

- From now on, we assume that there is a cross-visibility across every edge \((v_i, v_j)\) in \(G_i\) for all \(j\) except \(i + 1\) and \(i - 1\).
- Using the following lemma, all reflex vertices of \(F_i\) can be identified from \(G_i\).

**Lemma:** If \((v_{j-1}, v_{j+1})\) is not an edge in \(G_i\) and there is a cross-visibility across the edge \((v_i, v_j)\), then \(v_j\) is a reflex vertex in \(F_i\).
Our algorithm for computing the maximum clique in $G_i$ needs the property that $v_i$ is convex in $F_i$.

**Lemma:** If $G_i$ is not the visibility graph of a convex fan, $G_i$ can be decomposed into subgraphs such that (i) each subgraph is the visibility graph of a convex fan and (ii) one of these subgraphs contains the maximum clique of $G_i$.

- From now on, we assume that $(v_{i-1}, v_{i+1})$ is an edge in $G_i$.
- Therefore, $F_i$ is a convex fan as $v_i$ is a convex vertex in $F_i$. 
Convexity checking

The problem now is to compute the maximum clique in the visibility graph $G_i$ of a convex fan $F_i$.

Check the convexity at every vertex $v_k$ for every pair of incoming and outgoing edges at $v_k$ and locate the valid pairs of edges at $v_k$.

Assigning weights to edges of $G_i$ using valid pairs of edges.
Lemma: The maximum among weights on edges of $G_i$ is the size of the maximum clique in $G_i$.

Lemma: The maximum clique in the visibility graph $G_i$ of a convex fan can be computed in $O(e)$ time.

Theorem: Given the visibility graph $G$ of a simple polygon $P$ and the Hamiltonian cycle in $G$ that corresponds to the boundary of $P$, the maximum clique in $G$ can be computed in $O(n^2e)$ time, where $n$ and $e$ are the number of vertices and edges of $G$. 

Weight assignments
Open problem

Given the visibility graph $G$ of a simple polygon, compute the maximum clique in $G$ without any additional information.
Thank you.