Computing the maximum clique in visibility graphs

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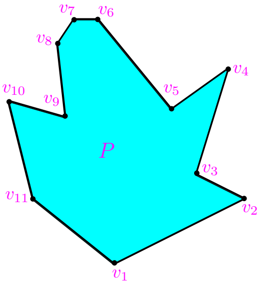
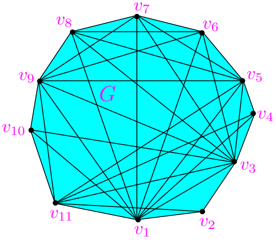
Overview

1. What is visibility graph?
2. Minimum dominating set and vertex guard problems.
3. Maximum hidden set problem.
4. Recognition problem.
5. Characterization problems.
6. Maximum clique problem.
7. Recognition and characterization problems.
8. An algorithm for computing a maximum clique.
9. Open problem

What is visibility graph?





- Let P be a polygon with or without holes.
- Two points u and w of P are said to be *visible* if the line segment uv is lies totally inside P.
- Construct a graph G from P such that every vertex of P is represented as a node in G and two nodes are connected in G iff their corresponding vertices in P are visible from each other.
- The graph G is called the *visibility graph* of P.

1. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, 2007.

Minimum dominating set and vertex guard problems

- The minimum dominating set problem in a visibility graph G is to locate a set S of nodes of G of the smallest size such that every node in G is connected by an edge in G to some vertex of S.
- The corresponding problem for a polygon P with or without holes is to locate a set S' of vertices of P of the smallest size such that every vertex of P is visible from some vertex of P (Art Gallery problem).
- Lin and Skiena showed that the problems of finding a minimum vertex cover and a maximum dominating set in the visibility graph of a simple polygon are NP-hard.

1. S. Y. Lin and S. Skiena, *Complexity aspects of visibility graphs*, International Journal of Computational Geometry and Applications, vol. 5, pp. 289-312, 1995.

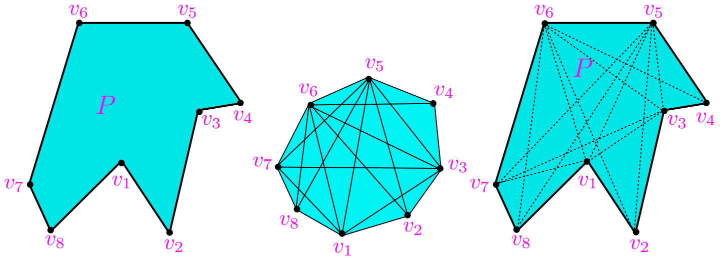
Maximum hidden set problem

1. The maximum independent set or hidden set problem in a visibility graph G is to locate a set S of nodes of G of the largest size such that no two nodes in S are connected by an edge in G.
2. Shermer showed that the maximum hidden set problem is NP-hard.
3. Hidden set has also been studied by other researchers.

1. T. Shermer, *Hiding People in Polygons*,Computing, vol. 42, pp. 109-131, 1989.
2. S. Eidenbenz, *In-approximability of visibility problems on polygons and terrains*, Ph. D. Thesis, Institute for Theoretical Computer Science, ETH, Zurich, 2000.
3. S. Eidenbenz, *In-approximability of finding maximum hidden sets on polygons and terrains*, Computational Geometry: Theory and Applications, vol. 21, pp. 139-153, 2002.

4. S. K. Ghosh, A. Maheshwari, S. P. Pal, S. Saluja and C. E. Veni Madhavan, *Characterizing and recognizing weak visibility polygons*, Computational Geometry: Theory and Applications, vol. 3, pp. 213-233, 1993.
5. S. K. Ghosh, T. Shermer, B. Bhattacharya and P. Goswami, *Computing the maximum clique in the visibility graph of a simple polygon*, Journal of Discrete Algorithms, vol. 5, pp. 524-532, 2007.
6. S. Y. Lin and S. Skiena, *Complexity aspects of visibility graphs*, International Journal of Computational Geometry and Applications, vol. 5, pp. 289-312, 1995.

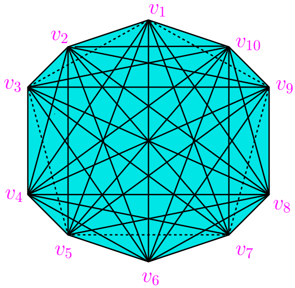
Recognition problem

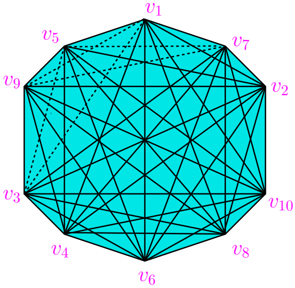


Let G be a graph. The problem of determining, if there is some polygon P that has G as its visibility graph, is called the *visibility graph recognition problem*.

Observe that the boundary of P corresponds to a Hamiltonian cycle of G.

The given graph G can have several Hamiltonian cycles.





The given graph G is not the visibility graph of any polygon P if the boundary of P corresponds to the Hamiltonian cycle $(v_1, v_5, v_9, v_3, v_4, v_6, v_8, v_{10}, v_2, v_7)$.

Problem: Given an undirected graph G with a Hamiltonian cycle, the problem of recognizing visibility graphs is to test whether there exists a simple polygon such that

1. the Hamiltonian cycle of the graph forms the boundary of a simple polygon,
2. two vertices of the simple polygon are visible if and only if they correspond to adjacent vertices in the graph.

1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Report JHU/E86/14, The Johns Hopkins University, 1986. Also in Lecture Notes in Computer Science, Springer-Verlag, pp. 96-104, 1988.
2. H. Everett, *Visibility graph recognition*, Ph. D. Thesis, University of Toronto, 1990.
3. G. Srinivasaraghavan and A. Mukhopadhyay, *A new necessary condition for the vertex visibility graphs of simple polygons*, Discrete and Computational Geometry, 12 (1994), 65–82.
4. J. Abello, K. Kumar and S. Pisupati, *On visibility graphs of simple polygons*, Congressus Numeratium, 90 (1992), 119-128.

4. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.
5. I. Streinu, *Non-stretchable pseudo-visibility graphs*, Computational Geometry:Theory and Applications, 31 (2005), 195-206.
6. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, 2007.

Characterization problem

Characterizing visibility graphs of simple polygons is an open problem.

Ghosh tried to characterize visibility graphs in term of perfect graphs, circle graphs or chordal graphs.

Coullard and Lubiw used clique ordering properties in their attempt to characterize visibility graphs.

1. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Report JHU/E86/14, The Johns Hopkins University, 1986. Also in Lecture Notes in Computer Science, Springer-Verlag, pp. 96-104, 1988.
2. S. K. Ghosh, *On recognizing and characterizing visibility graphs of simple polygons*, Discrete and Computational Geometry, 17 (1997), 143–162.
3. C. Coullard and A. Lubiw, *Distance Visibility graphs*, International Journal of Computational Geometry and Applications, 2 (1992), 349–362.

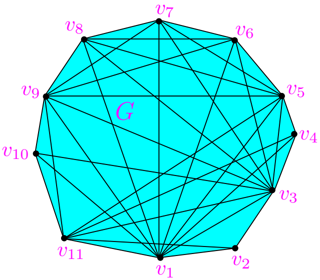
Everett and Corneil attempted to characterize visibility graphs using forbidden induced sub-graphs.

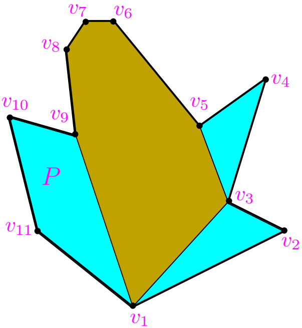
Abello and Kumar attempted to characterize visibility graphs using Euclidean shortest paths.

Open problem: In-spite of several attempts by many researchers in the last three decades, the problems of recognizing, characterizing and reconstructing visibility graphs of simple polygons are still open.

1. H. Everett, *Visibility graph recognition*, Ph. D. Thesis, University of Toronto, 1990.
2. H. Everett and D. G. Corneil, *Negative results on characterizing visibility graphs*, Computational Geometry: Theory and Applications, 5 (1995), 51-63.
3. J. Abello and K. Kumar, *Visibility Graphs and Oriented Matroids*, Discrete and Computational Geometry, 28 (2002), 449-465.

Maximum clique problem





- The *maximum clique* in G is a complete subgraph of G having the maximum number of vertices.
- Vertices v_1, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}, and v_{11} have formed a maximum clique in G.
- The maximum clique in G corresponds to the largest empty convex polygon in P that can be inscribed inside P with the maximum number of vertices of P.

Previous results

- The problem of computing the maximum clique in a visibility graph G of a polygon P with holes is known to be NP-hard.
- It is not clear whether the problem is NP-hard for visibility graphs G of a set disjoint line segments.
- The problem is not known to be NP-hard if the given graph G is the visibility graph of a simple polygon.
- On the other hand, there is no polynomial time algorithm for computing a maximum clique in the visibility graph G of a simple polygon P.

1. S. Eidenbenz and C. Stamm, *Maximum clique and minimum clique partition in visibility graphs*, Proceedings of IFIP TCS, Lecture Notes in Computer Science, vol. 1872, pp. 200- 212, 2000.

- Given a set S of n points, the largest empty convex polygon in S can be computed in in $O(n^3)$ by the algorithm of Avis and Rappaport.
- If the visibility graph G and its corresponding polygon P are given, the maximum clique in G of P can be computed in $O(n^3)$ time by the algorithm of Eidenbenz and C. Stamm.

1. D. Avis and D. Rappaport, *Computing the largest empty convex subset of a set of points*, Proceedings of the 1st ACM Symposium on Computational Geometry, pp. 161-167, 1985.
2. S. Eidenbenz and C. Stamm, *Maximum clique and minimum clique partition in visibility graphs*, Proceedings of IFIP TCS, Lecture Notes in Computer Science, vol. 1872, pp. 200- 212, 2000.

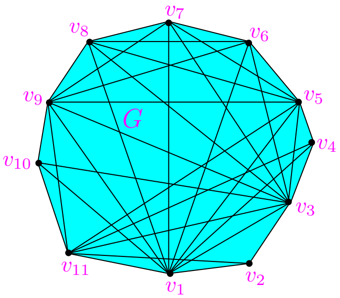
- Can we construct the corresponding simple polygon P of a given visibility graph G and then compute the maximum clique in G?
- However, there is no polynomial time algorithm known for constructing the corresponding simple polygon P from a given visibility graph G.
- Therefore, it is not possible to compute the maximum clique in G without having any geometric information of the corresponding polygon P.

With additional information

- Assume that the Hamiltonian cycle C in the visibility graph G corresponding to the boundary of P is given along with G.
- With this additional information, the maximum clique in G can be computed in $O(n^2 e)$ time by the algorithm of Ghosh et al., where n and e are number of vertices and edges in G respectively.
- Note that there is no polynomial time algorithm known for locating the Hamiltonian cycle C in G that corresponds to the boundary of P.

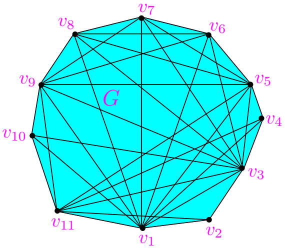
1. S. K. Ghosh, T. Shermer, B. Bhattacharya and P. Goswami, *Computing the maximum clique in the visibility graph of a simple polygon*, Journal of Discrete Algorithms, vol. 5, pp. 524-532, 2007.
2. S. K. Ghosh, Visibility algorithms in the plane, Cambridge University Press, Cambridge, 2007.

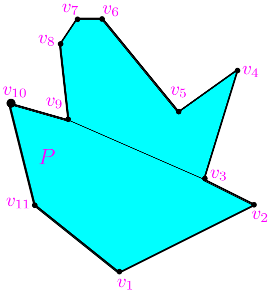
Decomposition of G in G_1, G_2,..., G_n



- For each vertex v_i in G, let G_i denote the induced subgraph of G formed by v_i and the neighbors of v_i in G.
- For example, G_3 is the entire visibility graph G.
- Observe that if v_i is a vertex of the maximum clique in G, then all vertices of the maximum clique in G belong to G_i.
- Our algorithm computes the maximum clique in each G_i and then chooses the one which is the largest.

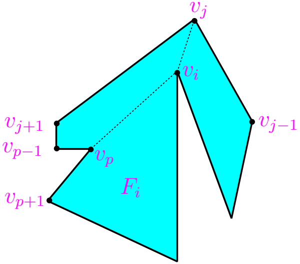
Fan vertex

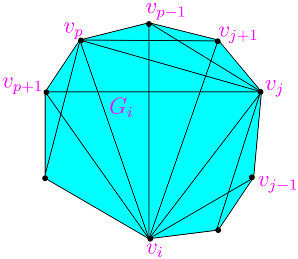




- The region of P, whose visibility graph is G_i, is called the *fan* F_i with v_i as the *fan vertex*.
- For example, the boundary of F_{10} consists of v_{10}, v_{11}, v_1, v_2, v_3, v_9 and v_{10}.
- Observe that the order of vertices on the boundary of F_i follows the order of vertices in C.
- If the internal angle θ_i at v_i in F_i is convex (i.e., $\theta_i \leq 180^\circ$), F_i is called a *convex fan*.
- Can we identify those vertices of G_i that are reflex in F_i?

Partition using cross-visibility



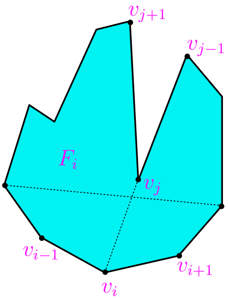


- The vertex v_j is not reflex in F_i as v_i is used to block the visibility between v_{j-1} and v_{j+1}.
- There is no cross-visibility across (v_i, v_j) as well as across (v_i, v_p).

Lemma: The graph G_i can be partitioned recursively into subgraphs using cross-visibility such that the maximum clique in G_i belongs to one of the subgraphs.

1. S. K. Ghosh, On recognizing and characterizing visibility graphs of simple polygons, Discrete and Computational Geometry, vol. 17, pp. 143-162, 1997.

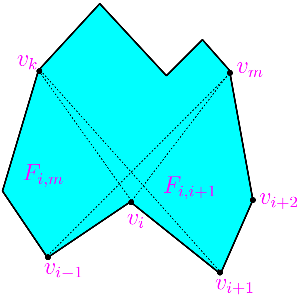
Identification of reflex vertices



- From now on, we assume that there is a cross-visibility across every edge (v_i, v_j) in G_i for all j except $i+1$ and $i-1$.
- Using the following lemma, all reflex vertices of F_i can be identified from G_i.

Lemma: If (v_{j-1}, v_{j+1}) is not an edge in G_i and there is a cross-visibility across the edge (v_i, v_j), then v_j is a reflex vertex in F_i.

Convex fan partition

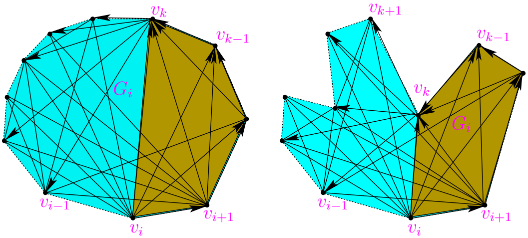


- Our algorithm for computing the maximum clique in G_i needs the property that v_i is convex in F_i.

Lemma: If G_i is not the visibility graph of a convex fan, G_i can be decomposed into subgraphs such that (i) each subgraph is the visibility graph of a convex fan and (ii) one of these subgraphs contains the maximum clique of G_i.

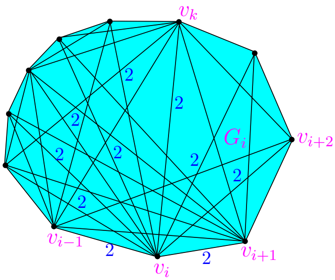
- From now on, we assume that (v_{i-1}, v_{i+1}) is an edge in G_i.
- Therefore, F_i is a convex fan as v_i is a convex vertex in F_i.

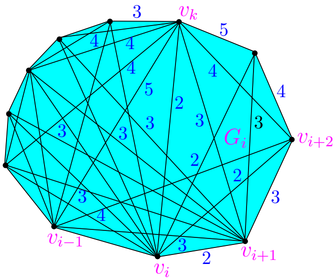
Convexity checking



- The problem now is to compute the maximum clique in the visibility graph G_i of a convex fan F_i.
- Check the convexity at every vertex v_k for every pair of incoming and outgoing edges at v_k and locate the valid pairs of edges at v_k.
- Assigning weights to edges of G_i using valid pairs of edges.

Weight assignments





Lemma: The maximum among weights on edges of G_i is the size of the maximum clique in G_i.

Lemma: The maximum clique in the visibility graph G_i of a convex fan can be computed in $O(e)$ time.

Theorem: Given the visibility graph G of a simple polygon P and the Hamiltonian cycle in G that corresponds to the boundary of P, the maximum clique in G can be computed in $O(n^2e)$ time, where n and e are the number of vertices and edges of G.

Open problem

Given the visibility graph G of a simple polygon, compute the maximum clique in G without any additional information.

Thank you.