Visibility Graph Theory for Points

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- Colouring Visibility Graphs
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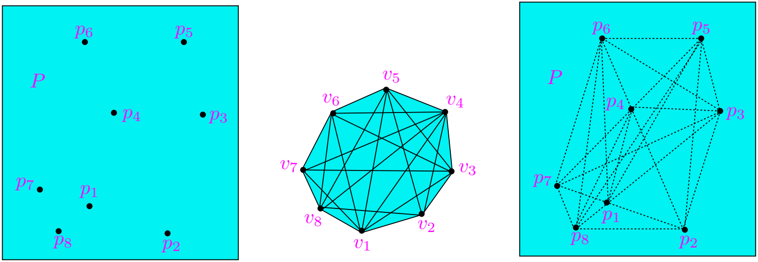
1. S. K. Ghosh and P. P. Goswami, *Unsolved problems in visibility graphs of points, segments and polygons*, manuscript, 2010. (Available at http://arxiv.org/abs/1012.5187).

Background



Let P be a set of n points in the plane.

Two points p_i and p_j of P are *mutually visible* if the line segment $p_i p_j$ does not contain or pass through any other point of P.

In other words, p_i and p_j are visible if $P \cap \overline{p_i p_j} = \{p_i, p_j\}$.

If a point $p_k \in P$ lies on the segment $p_i p_j$ connecting two points p_i and p_j in P, we say that p_k blocks the visibility between p_i and p_j, and p_k is called a *blocker* in P.

The *visibility graph* (also called the *point visibility graph*) G of P is defined by associating a vertex v_i with each point p_i of P such that (v_i, v_j) is an undirected edge of G if p_i and p_j are mutually visible.

If no three points of P are collinear, i.e., there is no blocker in P, then G is a complete graph as each pair of points in P is visible.

The visibility graph G can be computed from P in $O(n^2)$ time.

1. B. Chazelle, L. J. Guibas and D.T. Lee, *The Power of Geometric Duality*, BIT, 25:76-90, 1985.
2. H. Edelsbrunner, J. O'Rourke and R. Seidel, *Constructing Arrangements of Lines and Hyperplanes with Applications*, SIAM Journal on Computing, 15:341-363, 1986.

Visibility Graphs: Recognition, Characterization, and Reconstruction

Consider the opposite problem of determining if there is a set of points P whose visibility graph is the given graph G. This problem is called the visibility graph *recognition* problem.

Open Problem 1: Given a graph G in adjacency matrix form, determine whether G is the visibility graph of a set of points P in the plane.

Identifying the set of properties satisfied by all visibility graphs is called the visibility graph *characterization* problem.

Open Problem 2: Characterize visibility graphs of point sets.

The problem of actually drawing one such set of points P whose visibility graph is the given graph G, is called the visibility graph *reconstruction* problem.

Open Problem 3: Given the visibility graph G of a set of points, draw the points in the plane whose visibility graph is G.

Colouring Visibility Graphs

Consider the problem of colouring the visibility graph $G = (V, E)$ of a point set P.

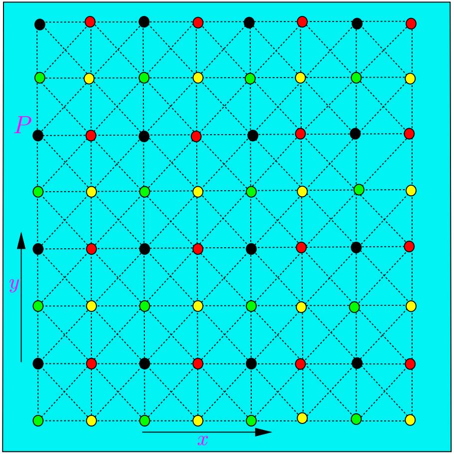
A k-colouring of G is a function $f : V \to C$ for some set C of k colours such that $f(v) \neq f(w)$ for every edge $(v_i, v_j) \in E$.

If G can be coloured by k colours, G is called *k-colourable*.

The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colourable.

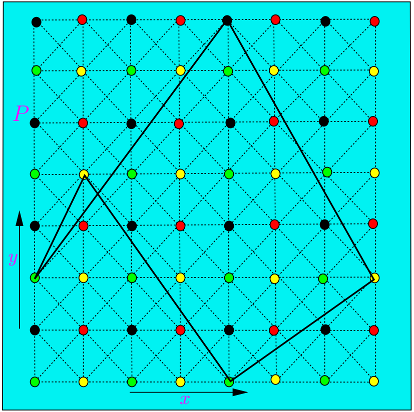
The *clique number* $\omega(G)$ is the maximum m such G contains a complete graph of m vertices as a subgraph.

1. J. Kŕa, A. Pór and D. R. Wood, *On the Chromatic Number of the Visibility Graph of a Set of Points in the Plane*, Discrete and Computational Geometry, 34(3):497-506, 2005.



Let $P = \{(x, y) : x, y \in \mathbb{Z}\}$ be the integer lattice. Then $\chi(G) = \omega(G) = 4$.

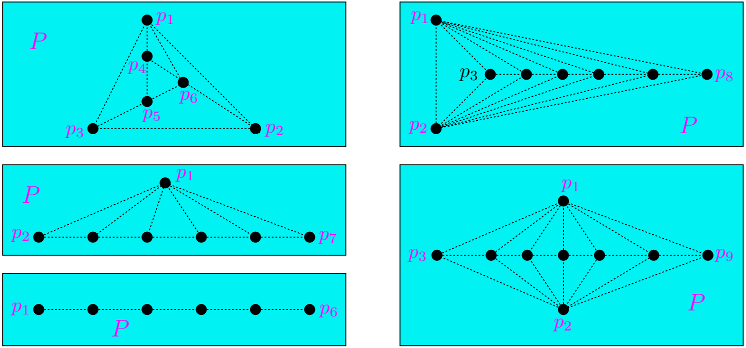
If a point set $P \subseteq \mathbb{R}^2$ can be covered by m lines, then $\chi(G) \leq 2m$.



In this graph, the chromatic number is same as the clique number but the graph is not a perfect graph as it contains a cycle of five vertices without chord.

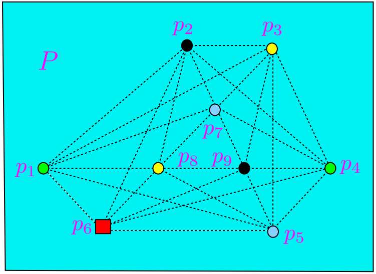
For example, the five lattice points with co-ordinates $(2,5)$, $(1,3)$, $(5,8)$, $(8,3)$, $(5,1)$ form a chordless cycle.

Conjecture: There exists a function f such that $\chi(G) \leq f(\omega(G))$.



Visibility graphs of points with $\omega(G) \leq 3$ are planar and they require at most 3 colours.

1. V. Dujmović, D. Eppstein, M. Suderman and D. R. Wood, *Drawings of planar graphs with few slopes and segments*, Computational Geometry: Theory and Applications, 38:194-212, 2007.



Visibility graphs with $\omega(G) = 4$ require 5 colours.

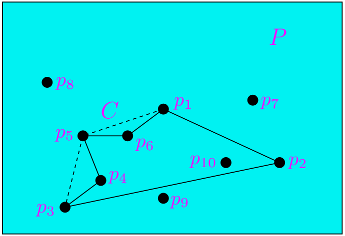
Open Problem 4: Prove that every visibility graph with $\omega(G) \leq 4$ has $\chi(G) \leq 5$.

Open Problem 5: Prove the conjecture for visibility graphs with $\omega(G) = 5$.

The conjecture does not hold for visibility graphs with $\omega(G) \geq 6$ as for every k, there is a finite point set $y \subset \mathbb{R}^2$ such that $\chi(G) \geq k$ and $\omega(G) = 6$.

1. F. Pfender, *Visibility graphs of point sets in the plane*, Discrete and Computational Geometry, 39(1):455-459, 2008.

Cliques in Visibility Graphs



Consider the situation where the visibility graph G of P is a complete graph, i.e., there is no three points of P ar collinier. So, $\omega(G)$ is the size of P.

Though the subgraph of G induced by any subset of vertices is a clique, the corresponding points $X \subseteq P$ may not form a convex polygon C.

Does there exists a smallest integer $g(k)$ for every positive integer k such that any point set P of at least $g(k)$ points in general position has a subset X of k points that are the vertices of a convex polygon C?

The existence of the value g(k) runs immediately from the famous Ramsey theorem.

The best known lower and upper bounds established for $g(k)$ are $2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 2$.

1. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Mathmatica, 2:463-470, 1935.
2. P. Erdös and G. Szekeres, *On some extremum problems in elementary geometry*, Annales Universitatis Scientarium Budapestinensis de Rolando Etvs Nominatae Sectio Mathematica, 3-4:53-61, 1961.
3. G. Tóth and P. Valtr, *The Erdös-Szekeres theorem: upper bounds and related results*, Combinatorial and Computational Geometry (Ed. J. E. Goodman, J. Pach and E. Welzl), 52:557-568, Cambridge University Press, Cambridge, 2005.

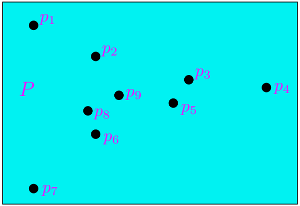
Even if all points of the subset X are in convex position forming a convex polygon C, some points of P may lie inside C.

Determining the smallest positive integer $h(k)$ (if it exists) such that any point set P of at least $h(k)$ points in general position in the plane has k points that are vertices of an empty convex polygon C.

For an empty triangle, $h(3) = 3$.

For an empty quadrilateral, it can be seen that $h(4) = 5$.

1. H. Harborth, *Konvexe Fnfecke in ebenen Punktmengen*, Elemente der Mathematik, 33:116-118, 1978.



There exists a set of 9 points such that no subset of 5 points forms an empty convex pentagon. So, $h(5) \geq 10$. In fact, $h(5) = 10$.

For an empty convex hexagon, two bounds are shown:
$h(6) \leq g(9) \leq 1717$ and $h(6) \leq g(25)$.

1. T. Gerken, *Empty convex hexagons in planar point sets*, Discrete and Computational Geometry, 39:239-272, 2008.
2. C. M. Nicolás, *The empty hexagon theorem*, Discrete and Computational Geometry, 38:389-397, 2007.

On the other hand, computer experiment shows that $h(6) \geq 30$.

The gap between the bounds has been reduced to
$h(6) \leq max\{g(8), 400\} \leq 463$.

For $k \geq 7$, $h(k)$ is not bounded.

1. M. H. Overmars, *Finding sets of points without empty convex 6-gons*, Discrete and Computational Geometry, 29:153-158, 2003.
2. V.A. Koshelev, *On the Erdös - Szekeres problem in combinatorial geometry*, Electronic Notes in Discrete Mathematics, 29: 175-177, 2007.
3. J.D. Horton, *Sets with no empty 7-gons*, Canadian Mathematical Bulletin, 26:482-484, 1983.

Consider the other situation where the visibility graph G of P is not a complete graph, i.e., there are collinear points in P.

So, the boundary of a convex polygon C formed by a subset of points X of P may contain collinear points.

For every integers $\ell \geq 2$ and $k \geq 3$, there exists a smallest integer $g(k, \ell)$ such that any point set P of at least $g(k, \ell)$ points in the plane contains (i) ℓ collinear points, or (ii) k points in strictly convex position.

1. Z. Abel, B. Ballinger, P. Bose, S. Collette, V. Dujmovic, F. Hurtado, S. Kominers, S. Langerman, A. Por and D. Wood, *Every large point set contains many collinear points or an empty pentagon*, Graphs and Combinatorics, 27(1):47-60, 2011.

A straightforward upper bound on $g(k, \ell)$ can be derived as follows:

Assume that P has $\ell - 1$ collinear points and at most $k - 1$ points in strictly convex position.

Let $X \subseteq P$ be any maximal set of points in strictly convex position.

Since every point of $P - X$ is collinear with two points in X, $\binom{|X|}{2}$ lines cover all points of P and each line can have at most $\ell - 3$ points of $P - X$.

Therefore, $|P| \leq \binom{|X|}{2}(\ell - 3) + |X| \leq \binom{k}{2}(\ell - 3) + k - 1$.

If one more point is added to P (i.e., $|P| \leq \binom{k}{2}(\ell - 3) + k)$), then P must contain ℓ collinear points or k points in strictly convex position.

A tighter upper bound on $g(k, \ell)$ has also been derived.

Though P with $g(k,\ell)$ points may have k points in strictly convex position, the convex polygon C formed by these k points may not be empty.

Therefore, the visibility graph G of P having $g(k,\ell)$ points may not have a clique of size k as some points of P lying inside C may block the visibility between vertices of C.

Conjecture: For all integers $k \geq 2$ and $\ell \geq 2$, there is an integer $h(k,\ell)$ such that any point set P of at least $h(k,\ell)$ points in the plane contains ℓ collinear points, or k mutually visible points.

1. J. Kŕa, A. Pór and D. R. Wood, *On the Chromatic Number of the Visibility Graph of a Set of Points in the Plane*, Discrete and Computational Geometry, 34(3):497-506, 2005.

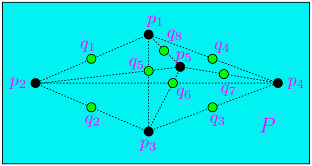
The conjecture is trivially true for $\ell \leq 3$ and for all k on any point set P having k points.

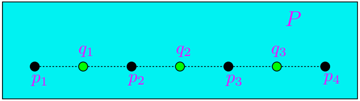
Every point set P of at least $max\{7, \ell + 2\}$ points contains ℓ collinear points or 4 mutually visible points.

The conjecture is also true for $k = 5$ and for all ℓ.

Open Problem 6: Prove the conjecture for $k = 6$ or $\ell = 4$.

Blockers of Visibility Graphs





Let P be a set of n points in the plane in general position.

Let $Q = (q_1, q_2, \ldots, q_j)$ be another set of points (or blockers) in the plane such that (i) $P \cap Q = \emptyset$ and (ii) every segment with both endpoints in P contains at least one point of Q. Any such set Q is called a *blocking set* for P.

Observe that (i) there is no edge in the visibility graph of $P \cup Q$ that connects two points of P and (ii) a blocker may block several pairs of visible points.

If all points of P are collinear, then $|P| - 1$ blockers are necessary and sufficient.

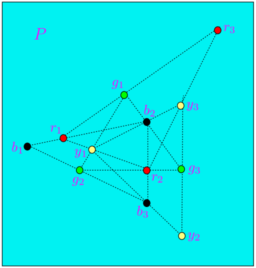
What is the minimum size $b(n)$ of blocking set Q for P?

It is obvious that $b(n) \geq n-1$. A better lower bound $b(n) \geq 2n-3$ follows form a triangulation of P. The bound has been improved to $b(n) \geq (\frac{25}{8} - o(1))n$.

Obvious upper bound on $b(n)$ is $b(n) \leq \binom{n}{2}$. The bound has been improved to $b(n) \leq n2^{c\sqrt{\log n}}$, where c is an absolute constant.

Open Problem 7: Prove that as $n \to \infty$, $\frac{b(n)}{n} \to \infty$.

1. A. Dumitrescu and J. Pach and G. Tóth, *A note on blocking visibility between points*, Geombinatorics, 19:67-73, 2009.
2. Y. V. Stanchescu, *Planar sets containing no three collinear points and non-averaging sets of integers*, Discrete Mathematics, 256:387-395, 2002.
3. J. Pach, *Midpoints of segments induced by a point set*, Geombinatorics, 13:98-105, 2003.

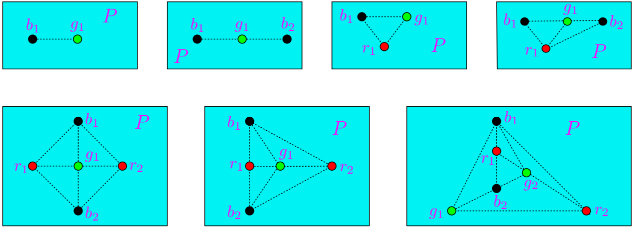


Let P be a set of points in the plane with some collinear points.

Assign $k \geq 2$ colours to points of P such that (i) if two points are mutually visible in P, assign different colours to them, and (ii) if two points are not visible due to some collinear points in P, assign the same colour to both of them.

Any set of points that admits such a colouring with k colours (for a fixed k) is called a *k-blocked point set*.

1. G. Aloupis, B. Ballinger, S. Collette, S. Langerman, A. Pór and D.R. Wood, *Blocking coloured point sets*, Proceedings of the 26th European Workshop Computational Geometry, pp. 29-32, 2010.



At most three points are collinear in every k-blocked point set.

Each colour class in a k-blocked point set is in a general position.

Conjecture: For each integer k, there is an integer n such that every k-blocked point set has at most n points.

Every 2-blocked point set has at most 3 points.

Every 3-blocked point set has at most 6 points.

Every 4-blocked point set has at most 12 points.

Conjecture: Every k-blocked point set has $O(k^2)$ points.

Conjecture: In every k-blocked point set, there are at most k points in each colour class.

Obstacle Representation of Visibility Graphs

Let $P = (p_1, p_2, \ldots, p_n)$ be a set of points in the plane.

Let $Q = (Q_1, Q_2, \ldots, Q_h)$ be a set of simple polygons in the plane called *obstacles*.

Construct the visibility graph G such that every point p_i of P is represented as a vertex v_i of G, and two vertices v_i and v_j of G are connected by an edge in G if and only if the line segment $p_i p_j$ does not intersect any obstacle Q_j for all j.

We call the pair (P, Q) as obstacle representation of G.

Given a visibility graph G, the problem of obstacle representation is to draw every vertex v_i of G as a point p_i in the plane and draw obstacles in such a way that every segment $p_i p_j$ intersects an obstacle if and only if (v_i, v_j) is not an edge in G.

The *obstacle number* is the minimum number of obstacles required in any obstacle representation of G, and is bounded above by $\binom{n}{2}$.

Open Problem 8: Is the obstacle number of a graph with n vertices bounded above by a linear function of n?

Open Problem 9: Improve the present lower bound $O(n/\log^2 n)$ of the obstacle number of a graph with n vertices.

Open Problem 10: For $h > 1$, what is the smallest number of vertices of a graph with obstacle number h?

Open Problem 11: Does every planar graph have obstacle number 1?

1. H. Alpert, C. Koch and J. D. Laison, *Obstacle numbers of graphs*, Discrete and Computational Geometry, 44:223–244, 2010.
2. P. Mukkamala, J. Pach, and Deniz Sarioz, *Graphs with large obstacle numbers*, Proceedings of the 36th International Workshop on Graph Theoretic Concepts in Computer Science, Lecture Notes in Computer Science, vol. 6410, pp. 292-303, Springer, 2010.

Concluding remarks

In this talk we have presented an overview of results on visibility graph theory for points and suggested several open problems. Hopefully, many more results will come which will enrich this fascinating area of geometric graph theory.