Approximation Algorithms for Art Gallery Problems in Polygons and Terrains

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Overview

1. Polygons and triangulations
2. Art gallery problems
3. Art gallery theorems on stationary guards
4. Art gallery theorems on mobile guards
5. Minimum number of guards
6. Approximation algorithms for vertex-guard problems
7. Approximation algorithms for edge-guard problems
8. Lower bounds on approximation ratios
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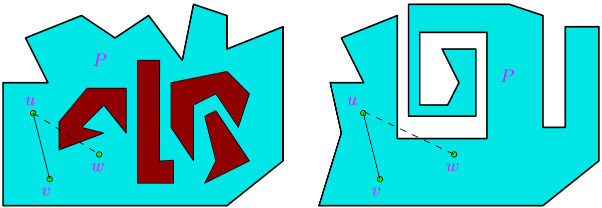
Polygon

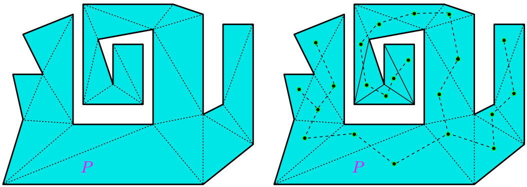


A *polygon* P is defined as a closed region in the plane bounded by a finite set of line segments (called *edges* of P) such that there exists a path between any two points of P which does not intersect any edge of P.

If the boundary of P consists of two or more cycles, then P is called a *polygon with holes*. Otherwise, P is called a *simple polygon* or a *polygon without holes*.

Two points u and v in a polygon P are said to be *visible* if the line segment joining u and v lies inside P.

Polygon triangulation



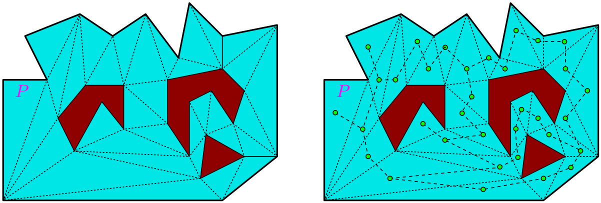
A line segment joining any two mutually visible vertices of a polygon is called a *diagonal* of the polygon.

Lemma: Every triangulation of a simple polygon P of n vertices uses $n-3$ diagonals and has $n-2$ triangles.

Corollary: The sum of the internal angles of a simple polygon of n vertices is $(n-2)\pi$.

Lemma: The dual of a triangulation of P is a tree.

Lemma: Every polygon P has at least two disjoint ears.

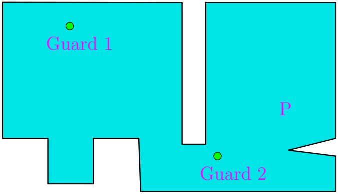


Lemma: Every triangulation of a polygon with h holes with a total of n vertices uses $n+3h-3$ diagonals and has $n+2h-2$ triangles.

Lemma: The dual graph of a triangulation of a polygon with holes must have a cycle.

1. G. H. Meisters, *Polygons have ears*, American Mathematical Monthly, 82 (1975), 648-651.
2. B. Chazelle, *Triangulating a simple polygon in linear time*, Discrete and Computational Geometry, 6 (1991), 485–524. Running time for triangulation is $O(n)$.
3. S. K. Ghosh and D. M. Mount, *An output sensitive algorithm for computing visibility graphs*, SIAM Journal on Computing, 20 (1991) 888-910. Running time for triangulation is $O(n \log n)$.
4. R. Bar-Yehuda, B. Chazelle, *Triangulating disjoint Jordan chains*, International Journal of Computational Geometry and Applications, 4 (1994) 475-481. Running time for triangulation is $O(n + h \log^{1+\epsilon} h)$, for $\epsilon > 0$.
5. J. O'Rourke, *Computational Geometry in C*, Cambridge University Press, 1994 (Second Edition in 1998).
6. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, Cambridge, UK, 2007.

Art gallery problem



An art gallery can be viewed as a polygon P with or without holes with a total of n vertices and guards as points in P.

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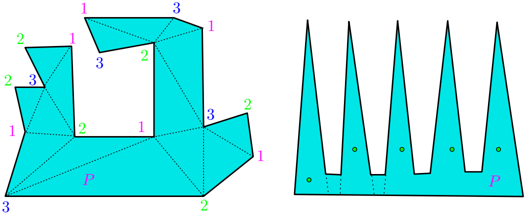
During a conference at Stanford in 1976, Victor Klee asked the following question:

How many guards are always sufficient to guard any polygon with n vertices?

1. R. Honsberger, *Mathematical games II*, Mathematical Associations for America, 1979.

Chvatal's art gallery theorem



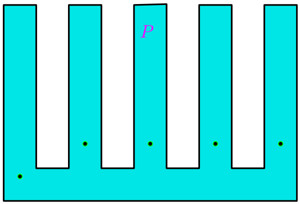
Theorem: A simple polygon P of n vertices needs at most $\lfloor \frac{n}{3} \rfloor$ guards.

Lemma: All vertices of P can be coloured using three colours (say, $\{1, 2, 3\}$) such that two vertices joined by an edge of P or by a diagonal in the triangulation of P receive different colours.

1. V. Chvatal, *A combinatorial theorem in plane geometry*, Journal of Combinatorial Theory, Series B, 18 (1975), 39-41.
2. S. Fisk, *A short proof of Chvatal's watchman theorem*, Journal of Combinatorial Theory, Series B, 24 (1978), 374.

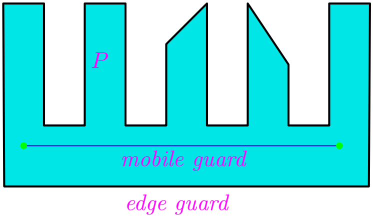
KKK's art gallery theorem

Theorem: An orthogonal polygon P of n vertices needs at most $\lfloor \frac{n}{4} \rfloor$ guards.



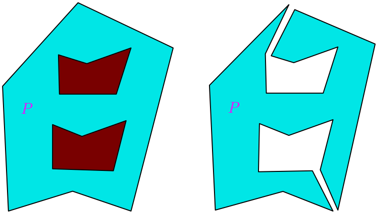
1. J. Kahn, M. Klawe and D. Kleitman, *Traditional galleries require fewer watchmen*, SIAM Journal of Algebraic and Discrete Methods, 4 (1983), 194-206.
2. J. O'Rourke, *An alternative proof of the rectilinear art gallery theorem*, Journal of Geometry, 211 (1983), 118-130.

Different types of guards



- *Point guards*: These are guards that are placed anywhere in the polygon.
- *Vertex guards*: These are guards that are placed on vertices of the polygon.
- *Edge guards*: These are guards that are allowed to patrol along an edge of the polygon.
- *Mobile guards*: These are guards that are allowed to patrol along a segment lying inside a polygon.

Art Gallery theorems in polygons with holes

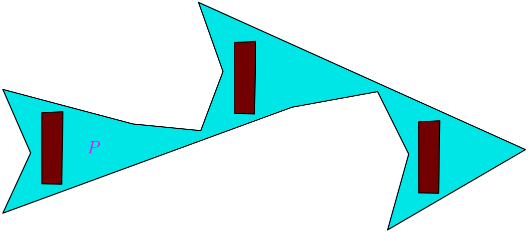


Theorem: Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards.

Conjecture: (Shermer) Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+h}{3} \rfloor$ vertex guards.

The conjecture has been proved by Shermer for $h = 1$. For $h > 1$, the conjecture is still open.

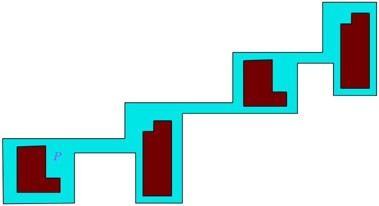
1. J. O'Rourke, *Art gallery theorems and algorithms*, Oxford University Press, 1987.



A polygon of 24 vertices with 3 holes requires 9 guards.

Theorem: To guard a polygon P with n vertices and h holes, $\lceil \frac{n+h}{3} \rceil$ point guards are always sufficient and occasionally necessary.

1. I. Bjorling-Sachs and D. L. Souvaine, *An efficient algorithm for guard placement in polygons with holes*, Discrete and Computational Geometry 13 (1995), 77-109.
2. F. Hoffmann, M. Kaufmann and K. Kriegel, *The art gallery theorem for polygons with holes*, Proceedings of the 32nd Symposium on the Foundation of Computer Science, 39-48, 1991.

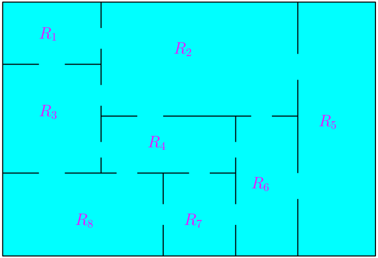


A polygon of 44 vertices with 4 holes requires 12 vertex guards.

Theorem: To guard an orthogonal polygon P with n vertices and h holes, $\lfloor \frac{n}{4} \rfloor$ point guards are always sufficient.

Theorem: To guard an orthogonal polygon P with n vertices and h holes, $\lfloor \frac{n}{3} \rfloor$ vertex guards are always sufficient.

1. F Hoffmann, *On the rectilinear art gallery problem*, Proceedings of ICALP, Lecture Notes in Computer Science, 443 (1990), 717-728, Springer-Verlag.
2. F. Hoffmann and K. Kriegel, *A graph-coloring result and its consequences for polygon-guarding problems*, SIAM Journal on Discrete Mathematics, 9 (1996), 210-224.



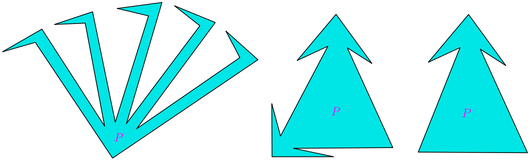
Theorem: Any rectangular art gallery with n rooms can be guarded with exactly $\lceil \frac{n}{2} \rceil$ guards.

1. J. Czyzowicz, E. Rivera-Campo, N. Santoro, J. Urrutia and J. Zaks, *Guarding rectangular art galleries*, Discrete Applied Mathematics, 50 (1994), 149-157.

Art Gallery theorems on mobile guard

In 1981, Godfried Toussaint asked the following question:
How many edge/mobile guards are always sufficient to guard any polygon with n vertices?

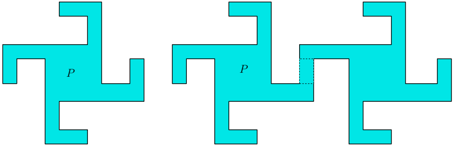
Theorem: To guard a simple polygon with n vertices, $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient and occasionally necessary.



Conjecture: (Toussaint) Except for a few polygons, $\lfloor \frac{n}{4} \rfloor$ edge guards are always sufficient to guard any simple polygon with n vertices.

1. J. O'Rourke, *Galleries need fewer mobile guards: A variation on Chvatal's theorem*, Geometricae Dedicata, 4 (1983), 273-283.

Theorem: To guard an orthogonal polygon with n vertices, $\lfloor \frac{3n+4}{16} \rfloor$ mobile or edge guards are always sufficient and occasionally necessary.



Theorem: To guard an orthogonal polygon with n vertices and h holes, $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are always sufficient and occasionally necessary.

1. A. Aggarwal, *The art gallery theorems: Its variations, applications and algorithmic aspects*, Ph. D. thesis, John Hopkins University, 1984.
2. I. Bjorling-Sachs, *Edge guards in rectilinear polygons*, Computational Geometry: Theory and Applications, 11 (1998), 111-123.
3. E. Gyri, F. Hoffmann, K. Kriegel and T. Shermer, *Generalized guarding and partitioning for rectilinear polygons*, CGTA, 6 (1996), 21-44.

Minimum number of guards

Let P be a polygon with or without holes. What is the minimum number of guards required for guarding a polygon P with or without holes?

Suppose, a positive integer k is given. Can it be decided in polynomial time whether k guards are sufficient to guard P?

The problem is NP-complete.

Theorem: The minimum vertex, point and edge guard problems for polygons with or without holes (including orthogonal polygons) are NP-hard.

Theorem: The minimum vertex and point guard problems for orthogonal polygons with or without holes are NP-hard.

1. D. T. Lee and A. K. Lin, *Computational Complexity of Art Gallery Problems*, IEEE Transactions on Information Theory, IT–32 (1986), 276–282.
2. J. O'Rourke and K. Supowit, *Some NP-hard polygon decomposition problems*, IEEE Transactions on Information Theory, IT-29 (1983), 181-190.
3. D. Schuchardt and H. Hecker, *Two NP-hard art-gallery problems for ortho-polygons*, Mathematical Logic Quarterly, 41 (1995), 261-267.
4. B.C. Liaw, N.F. Huang, R.C.T. Lee, *The minimum cooperative guards problem on k-spiral polygons*, Proceedings of the Canadian Conference on Computational Geometry, pp. 97–101, 1993.
5. M. Katz and G. Roisman, *On guarding the vertices of rectilinear domains*, Computational Geometry: Theory and Applications, 39 (2008), 219-228.

Approximation algorithms for minimum guard problems

An algorithm that returns sub-optimal solutions for an optimization problem is called an *approximation algorithm*.

Approximation algorithms are often associated with NP-hard problems; since it is unlikely that there can ever be efficient polynomial time exact algorithms solving NP-hard problems, one settles for polynomial time sub-optimal solutions.

We present polynomial time approximation algorithms for the minimum vertex and edge guard problems in a polygon P with or without holes.

1. S. K. Ghosh, *Approximation algorithms for art gallery problems*, Technical report no. JHU/EECS-86/15, Department of Electrical Engineering and Computer Science, The Johns Hopkins University, August 1986. Also in Proceedings of the Canadian Information Processing Society Congress, pp. 429-434, 1987. Running time: $O(n^5 \log n)$ time. Approximation ratio: $O(\log n)$.
2. S. K. Ghosh, *Approximation algorithms for art gallery problems in polygons*, Discrete Applied Mathematics, vol. 158, pp. 718-722, 2010 . The running time has been improved to $O(n^4)$ for simple polygons and $O(n^5)$ for polygons with holes, keeping the approximation ratio same.
3. A. Efrat and S. Har-Peled, *Guarding galleries and terrains*, Information Processing Letters, 100 (2006), 238-245. Running time and the approximation ratio: (i) For simple polygons, $O(nc_{opt}^2 \log^4 n)$ expected time, and $O(\log c_{opt})$ approximation ratio, where c_{opt} is the number of vertices in the optimal solution. (ii) For polygons with h holes, $O(nhc_{opt}^3 polylog\ n)$ expected time, $O(\log n \log(c_{opt} \log n))$ approximation ratio.

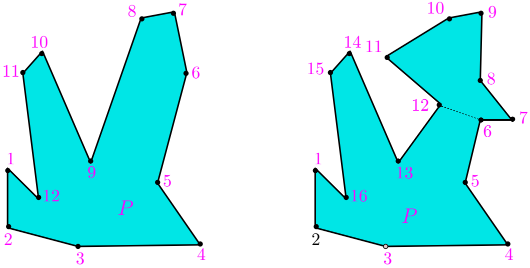
4. B. J. Nilsson, *Approximate Guarding of Monotone and Rectilinear Polygons*, Proceedings of the 32nd International Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science, Springer-Verlag, no. 3580, pp. 1362-1373, 2005. The paper gives polynomial time approximation algorithms (i) for monotone polygons and (ii) for simple orthogonal polygons. Approximation ratios are 12 and 96 respectively.
5. A. Deshpande, T. Kim, E. D. Demaine1 and S. E. Sarma, *A pseudopolynomial time O(log n)-approximation algorithm for art gallery problems* , Proceedings of the 10th International Workshop on Algorithms and Data Structures, LNCS, Springer-Verlag, no. 4619, pp. 163-174, 2007. Running time: Polynomial in n, the number of walls and the spread, where the spread can be exponential. Approximation ratio: $O(\log c_{opt})$.

Heuristics for Stationary Guard Problems

- Recently, efforts are being made to design heuristics to solve stationary guard problems where the efficiency of these heuristics are evaluated by experimentation.
- These heuristics essentially follow Ghosh's method of first discretizing the entire region of a polygon and then using the minimum set-cover solution.
- However, these heuristics use different criteria for discretization or in choosing candidate sets.

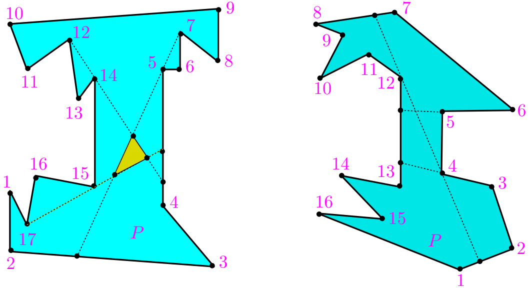
1. Y. Amit and J. S. B. Mitchell and E. Packer, *Locating Guards for Visibility Coverage of Polygons*, Proceedings of the 9th Workshop on Algorithm Engineering and Experiments (ALENEX'07), SIAM, pp. 120-134, 2007.
2. M. C. Couto and P. J. de Rezende and C. C. de Souza, *An exact and efficient algorithm for the orthogonal art gallery problem*, Proceedings of the 20th Brazilian Symposium on Computer Graphics and Image Processing, pp. 87-94, 2007.
3. M. C. Couto and P. J. de Rezende and C. C. de Souza, *Experimental evaluation of an exact algorithm for the orthogonal art gallery problem*, Proceedings of the 7th International Workshop on Experimental Algorithms, LNCS, vol. 5038, pp. 101-113, Springer, Heidelberg, 2008.
4. M. C. Couto and P. J. de Rezende and C. C. de Souza, An IP solution to the art gallery problem, Proceedings of the 25th Annual ACM Symposium on Computational Geometry, pp. 88-89, 2009.

Vertex-guard problem



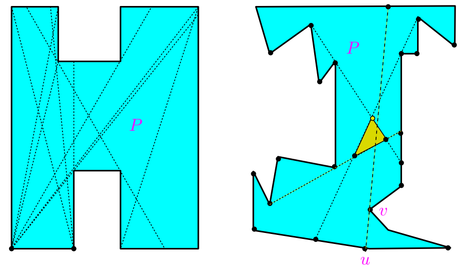
A simple polygon is called a *fan* if there exists a vertex that is visible from all points in the interior of the polygon.

The vertex guard problem can be treated as a polygon decomposition problem in which the decomposition pieces are fans.



Vertices 7, 12 and 17 together can see the entire boundary of the polygon but the shaded region is not visible from any of these vertices.

Three fans (vertices 1, 4 and 7) are necessary to cover the polygon if only edge extensions are allowed, whereas two fans (vertices 1 and 7) suffice if we allow the boundary of convex components to be bounded by segments that contains any two vertices of the polygon.

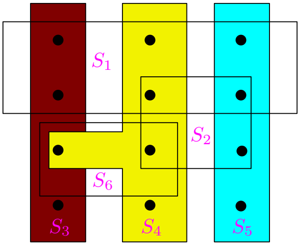


The polygonal region of P is decomposed into convex components where every component is bounded by segments that contains any two vertices of the polygon.

Every convex component must lie in at least one of the fans chosen by the approximation algorithm.

Lemma: Every convex component is either totally visible or totally invisible from a vertex of P.

Minimum set-covering problem



Minimum set-covering problem: Given a finite family C of sets $S_1, \ldots, S_n$, the problem is to determine the minimum cardinality $A \subseteq C$ such that $\bigcup_{i \in A} S_i = \bigcup_{j=1}^{n} S_j$.

The problem of finding the minimum number of fans to cover P is same as the minimum set-covering problem, where every fan is a set and convex components are elements of the set.

1. D. S. Johnson, *Approximation algorithms for combinatorial problems*, Journal of Computer and System Sciences, 9 (1974), 256-278.

Vertex-guard algorithm

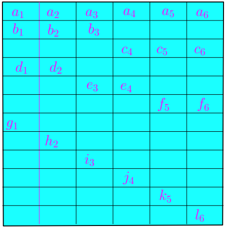
Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.
Step 2: *For* $1 \leq j \leq n$, construct the set F_j by adding those convex components of P that are totally visible from the vertex v_j.
Step 3: Find $i \in N$ such that $|F_i| \geq |F_j|$ *for all* $j \in N$ and $i \neq j$.
Step 4: Add i to Q and delete i from N.
Step 5: *For all* $j \in N$, $F_j := F_j - F_i$, and $C := C - F_i$.
Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.
Step 7: Output the set Q and Stop.

Theorem: The approximation algorithm for the minimum vertex guard problem in a polygon P of n vertices computes solutions that are at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

Edge-guard algorithm

Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.
Step 2: *For* $1 \leq j \leq n$, construct the set E_j by adding those convex components of P that are totally visible from the edge e_j of P.
Step 3: Find $i \in N$ such that $|E_i| \geq |E_j|$ *for all* $j \in N$ and $i \neq j$.
Step 4: Add i to Q and delete i from N.
Step 5: *For all* $j \in N$, $E_j := E_j - E_i$, and $C := C - E_i$.
Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.
Step 7: Output the set Q and Stop.
Theorem: For the minimum edge guard problem in an n-sided polygon P, an approximate solution can be computed which is at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

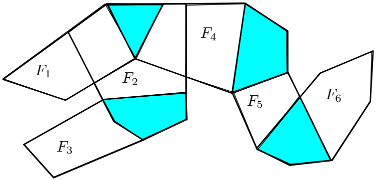
Upper bound on approximation ratio



Sets corresponding to rows are $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, $\{b_1, b_2, b_3\}$, $\{c_4, c_5, c_6\}$, $\{d_1, d_2\}$, $\{e_3, e_4\}$, $\{f_5, f_6\}$, $\{g_1\}$, $\{h_2\}$, $\{i_3\}$, $\{j_4\}$, $\{k_5\}$, $\{l_6\}$. Sets corresponding to columns are $\{a_1, b_1, d_1, g_1\}$, $\{a_2, b_2, d_2, h_2\}$, $\{a_3, b_3, e_3, i_3\}$, $\{a_4, c_4, e_4, j_4\}$, $\{a_5, c_5, f_5, k_5\}$, $\{a_6, c_6, f_6, l_6\}$.

Optimal cover chooses 6 sets corresponding to 6 columns (say, $k = 6$) but the greedy algorithm chooses sets corresponding to rows (i.e., $\frac{k}{6} + \frac{k}{3} + \frac{k}{2} + \frac{k}{1} < k \log k$).

Any set consisting of arbitrary chosen convex components may not form a fan as every fan consists of contiguous convex components. Therefore, constructing any example where the greedy algorithm takes $O(\log n)$ times optimal does not seem to be possible.



Conjecture: (Ghosh 1986) Approximation algorithms are expected to yield solutions within a constant factor of the optimal.

Lower bound on approximation ratio

Regarding the lower bound on the approximation ratio for the problems of minimum vertex, point and edge guards in simple polygons, it has been shown that these problems are APX-hard using gap-preserving reductions from 5-OCCURRENCE-MAX-3-SAT.

This means that for each of these problems, there exists a constant $\epsilon > 0$ such that an approximation ratio of $1 + \epsilon$ cannot be guaranteed by any polynomial time approximation algorithm unless $P = NP$.

The above statement implies that there may be approximation algorithms for these problems whose approximation ratios are not small constants.

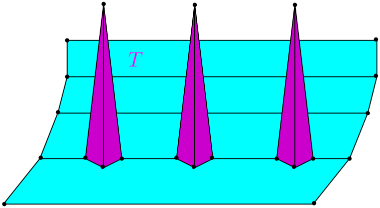
1. S. Eidenbenz C. Stamm and P. Widmayer, *Inapproximability Results for Guarding Polygons and Terrains*, Algorithmica, 31 (2000), 79-113.

On the other hand, for polygons with holes, these problems cannot be approximated by a polynomial time algorithm with ratio $((1-\epsilon)/12)(\mathit{ln}\ n)$ for any $(\epsilon > 0)$, unless $NP \subseteq TIME(n^{O(loglogn)})$. The results are obtained by using gap-preserving reductions from the SET COVER problem.

Open problems

Design approximation algorithms for vertex, edge and point guards problems in simple polygons which yield solutions within a constant factor of the optimal.

Guarding 2.5-dimensional terrains



Let T denote a polyhedral surface such that any vertical line intersects T exactly at one point. Then, T is called *2.5-dimensional terrain*.

Theorem: For guarding T of n vertices,
(i) $\lfloor \frac{n}{2} \rfloor$ vertex guards are both necessary and sufficient,
(ii) $\lfloor \frac{n}{3} \rfloor$ edge guards are always sufficient, and
(iii) $\lfloor \frac{4n-4}{13} \rfloor$ edge guards are sometime necessary.

1. P. Bose, T. Shermer, G. T. Toussaint and B. Zhu, *Guarding polyhedral terrains*, Computational Geometry: Theory and Applications, 7 (1997) 173-185.

The algorithms for placing $\lfloor \frac{n}{2} \rfloor$ vertex guards and $\lfloor \frac{n}{3} \rfloor$ edge guards in T use the technique of maximum matching in a bridgeless cubic graph and run in $O(n^{3/2})$ time.

1. H. Everett and Eduardo Rivera-Campo, *Edge Guarding Polyhedral Terrains*, Computational Geometry: Theory and Applications, 7 (1997), 201-203.
2. P. Bose, D. G. Kirkpatrick and Z. Li, *Efficient algorithms for guarding or illuminating the surface of a polyhedral terrain*, In Proceedings of the 8th Canadian Conference on Computational Geometry, pp. 217–222, 1996.

Minimum guard problems

Theorem: The minimum point guard problem for T is NP-hard.

There is an approximation algorithm for the minimum vertex guard problem in T.

Open problems

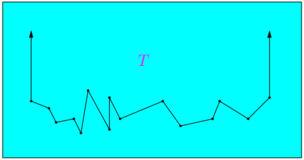
(i) Prove that the minimum vertex and edge guard problems for T are also NP-hard.

(ii) Design approximation algorithms for the minimum point and edge guard problems for T.

1. R. Cole and M. Sharir, *Visibility problems for polyhedral terrains*, Journal of Symbolic Computation, 7 (1989), 11-30.
2. S. Eidenbenz, *Approximation algorithms for terrain guarding*, Information Processing Letters, 82 (2002), 99-105.

Running time: $O(n^8)$ time. Approximation ratio: $O(\log n)$.

Guarding 1.5-dimensional terrains



Let C denote a polygonal chain which is monotone with respect to X-axis. The region of the plane lying above C is called *1.5-dimensional terrain*.

The minimum vertex guard problem for T is NP-hard.

1. D. Chen, V. Estivill-Castro and J. Urrutia, *Optimal guarding of polygons and monotone chains*, In Proceedings of the 7th Canadian Conference on Computational Geometry, pp. 133–138, 1995. There is a gap in the proof of the hardness claim in the paper.
2. J. King and E. Krohn, *Terrain guarding is NP-Hard*, Proceedings of the ACM-SIAM Symposium on Discrete Algorithms, pp. 1580- 1593, 2010.

Approximation algorithms for guarding 1.5-dimensional terrains

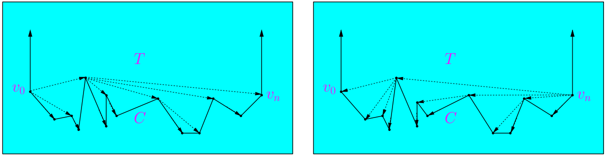
1. B. Ben-Moshe, M. Katz and J. Mitchell, *A constant-factor approximation algorithm for optimal terrain guarding*, SIAM Journal on Computing, 36 (2007), 1631-1647. Running time: $O(n^4)$ time. Approximation ratio: $O(1)$.
2. K. L. Clarkson and K. R. Varadarajan, *Improved approximation algorithms for geometric set cover*, Discrete & Computational Geometry, 37 (2007), 43-58. Running time: $O(n^2 \log n)$ time. Approximation ratio: $O(1)$.
3. J. King, *A 4-approximation algorithm for guarding 1.5-dimensional terrain*, LATIN 2006, LNCS, Springer-Verlag, no. 3887, pp. 629-640, 2007. Running time: $O(n^2)$ time. Approximation ratio: 5.

5. K. Elbassioni, D. Matijevic, J. Mestre and D. Severdija, *Improved approximations for guarding 1.5-dimensional terrains*, Algorithmica, 2010 (to appear). Running time: $O(n \log n)$ time. Approximation ratio: 4.

6. M. Katz and G. Roisman, *On guarding the vertices of rectilinear domains*, Computational Geometry: Theory and Applications, 39 (2008), 219-228. Running time: $O(n^2)$ time for guarding 1.5-dimensional rectilinear terrains. Approximation ratio: 2.

7. M. Gibson and G. Kanade and E. Krohn and K. Varadarajan, *An approximation scheme for terrain guarding*, Proceedings of the 12th. International Workshop on Approximation Algorithms for Combinatorial Optimization Problems, LNCS, Springer, vol. 5687, pp. 140-148, The polynomial time algorithm returns a guard cover whose cardinality is at most $(1+\epsilon)$ time optimal for any $\epsilon > 0$.

Guarding algorithm of Elbassioni et al.



- Number the vertices of the polygonal chain as $v_0, v_1, \ldots, v_n$ from left to right.
- A guard g place on a vertex v_i is called a *left vertex guard* of a vertex v_j if $i < j$ and v_i and v_j be two mutually visible vertices of C. A *right vertex guard* of v_j is defined analogously.
- Place left guards at all vertices of the Euclidean shortest path tree rooted at v_0 which are not leaves of the tree.
- Place right guards at all vertices of the Euclidean shortest path tree rooted at v_n which are not leaves of the tree.

1. K. Elbassioni, D. Matijevic, J. Mestre and D. Severdija, *Improved approximations for guarding 1.5-dimensional terrains*, Algorithmica, 2010 (to appear).

- Every vertex of C is now guarded by both left and right guards.
- In order to minimize the number of guards, vertices are partitioned into two sets L and R such that if a vertex v_j belongs to L (or R) then its left (respectively, right) vertex guard is added to the selected list of guards G.
- The partition of vertices in L and R is done using linear programming relaxation method.

Theorem: For a 1.5-dimensional terrain T of n vertices, an approximate solution of the minimum vertex guard problem in T can be computed in $O(n \log n)$ time and the size of the solution can be at most four times the optimal.

Concluding remarks

In this lecture, we have presented an overview of approximation algorithms that are designed for art gallery problems in polygons with or without holes, and in 1.5 and 2.5-dimensional terrains.

Thank you.