Old and New Algorithms for Guarding Polygons

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Abū Jafar Muhammad al-Khwārizmī, a Persian astronomer and mathematician, wrote a treatise in 825 AD, Kitāb hisāb al-adad al-hindī (Book on Calculation with Hindu Numerals), which was translated into Latin in the early 12th century as Liber Algorismi de numero Indorum (The Book of al-Khwārizmī on Indian Numerals). The word "Algorism"- the Latin form of al-Khwārizmī's name - came to be applied to any systematic work on ancient Indian-style computational mathematics. The present term "algorithm" is a distorted form of "algorism".

Overview

We present three algorithms from the following papers.

1. S. K. Ghosh, *Approximation algorithms for art gallery problems*, Technical report no. JHU/EECS-86/15, Department of Electrical Engineering and Computer Science, The Johns Hopkins University, August 1986. Also in Proceedings of the Canadian Information Processing Society Congress, pp. 429-434, 1987. Running time: $O(n^5 \log n)$. Approximation ratio: $O(\log n)$.

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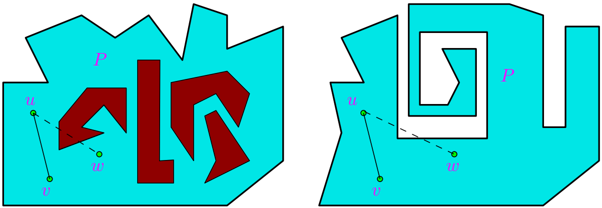
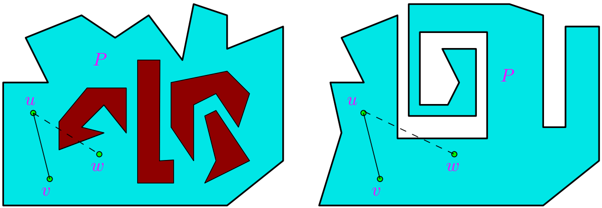
4. A. Baertschi, S. K. Ghosh, M. Mihalak, T. Tschager and P. Widmayer, *Improved bounds for the conflict-free chromatic art gallery problem*, Proceedings of the 30th ACM Annual Symposium on Computational Geometry, pp. 144-153, 2014. Running time: $O(n^2)$. Chromatic guard number: $O(\log n)$.

Visibility in Polygons



A *polygon* P is defined as a closed region in the plane bounded by a finite set of line segments (called *edges* of P) such that there exists a path between any two points of P which does not intersect any edge of P.

Visibility in Polygons



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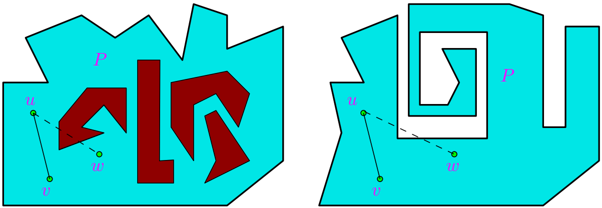

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Two points u and v in a polygon P are said to be *visible* if the line segment joining u and v lies entirely inside P.

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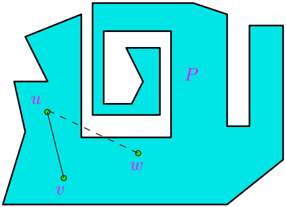

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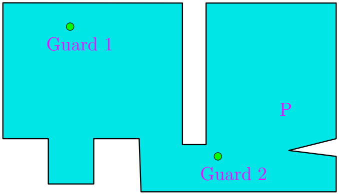
A polygon P is said to be a *weak visibility polygon* if every point in P is visible from some point of an edge uv.

1. S. K. Ghosh, *Visibility Algorithms in the Plane*, Cambridge University Press, Cambridge, UK, 2007.

Art gallery problem



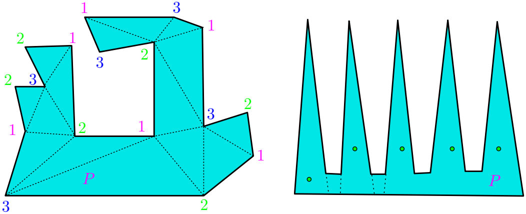
An art gallery can be viewed as a polygon P with or without holes with a total of n vertices and guards (or cameras) as points in P.

During a conference at Stanford in 1976, Victor Klee asked the following question:

How many guards are always sufficient to guard any polygon with n vertices?

1. R. Honsberger, *Mathematical games II*, Mathematical Associations for America, 1979.

Chvatal's art gallery theorem

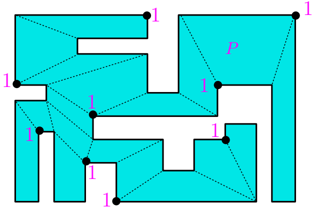


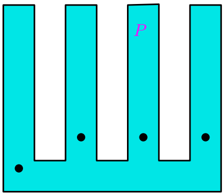
Theorem: A simple polygon P of n vertices needs at most $\lfloor \frac{n}{3} \rfloor$ guards.
Lemma: All vertices of P can be colored using three colors (say, $\{1, 2, 3\}$) such that two vertices joined by an edge of P or by a diagonal in the triangulation of P receive different colors.

1. V. Chvatal, *A combinatorial theorem in plane geometry*, Journal of Combinatorial Theory, Series B, 18 (1975), 39-41.
2. S. Fisk, *A short proof of Chvatal's watchman theorem*, Journal of Combinatorial Theory, Series B, 24 (1978), 374.

KKK's art gallery theorem

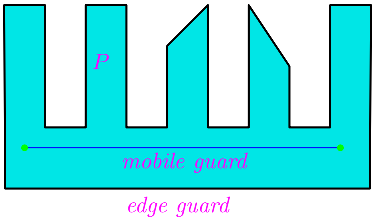
Theorem: An orthogonal polygon P of n vertices needs at most $\lfloor \frac{n}{4} \rfloor$ guards.





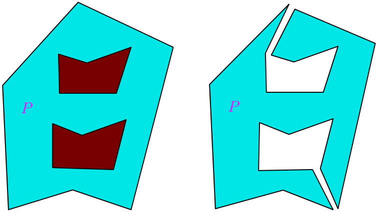
1. J. Kahn, M. Klawe and D. Kleitman, *Traditional galleries require fewer watchmen*, SIAM Journal of Algebraic and Discrete Methods, 4 (1983), 194-206.
2. J. O'Rourke, *An alternative proof of the rectilinear art gallery theorem*, Journal of Geometry, 211 (1983), 118-130.

Different types of guards



- *Point guards*: These are guards that are placed anywhere in the polygon.
- *Vertex guards*: These are guards that are placed on vertices of the polygon.
- *Edge guards*: These are guards that are allowed to patrol along an edge of the polygon.
- *Mobile guards*: These are guards that are allowed to patrol along a segment lying inside a polygon.

Art Gallery theorems in polygons with holes



Theorem: Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards.

Conjecture: (Shermer) Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+h}{3} \rfloor$ vertex guards.

The conjecture has been proved by Shermer for $h = 1$. For $h > 1$, the conjecture is still open.

1. J. O'Rourke, *Art gallery theorems and algorithms*, Oxford University Press, 1987.

Minimum number of guards

Let P be a polygon with or without holes. What is the minimum number of guards required for guarding a polygon P with or without holes?

Suppose, a positive integer k is given. Can it be decided in polynomial time whether k guards are sufficient to guard P?

The problem is NP-complete.

Theorem: The minimum vertex, point and edge guard problems for polygons with or without holes (including orthogonal polygons) are NP-hard.

Theorem: The minimum vertex and point guard problems for orthogonal polygons with or without holes are NP-hard.

1. D. T. Lee and A. K. Lin, *Computational Complexity of Art Gallery Problems*, IEEE Transactions on Information Theory, IT–32 (1986), 276–282.
2. A. Aggarwal, *The art gallery theorems: Its variations,applications and algorithmic aspects*, Ph. D. thesis, John Hopkins University, 1984.
3. J. O'Rourke and K. Supowit, *Some NP-hard polygon decomposition problems*, IEEE Transactions on Information Theory, IT-29 (1983), 181-190.
4. D. Schuchardt and H. Hecker, *Two NP-hard art-gallery problems for ortho-polygons*, Mathematical Logic Quarterly, 41 (1995), 261-267.
5. B.C. Liaw, N.F. Huang, R.C.T. Lee, *The minimum cooperative guards problem on k-spiral polygons*, Proceedings of the Canadian Conference on Computational Geometry, pp. 97–101, 1993.
6. M. Katz and G. Roisman, *On guarding the vertices of rectilinear domains*, Computational Geometry: Theory and Applications, 39 (2008), 219-228.

Approximation algorithms for minimum guard problems

An algorithm that returns sub-optimal solutions for an optimization problem is called an *approximation algorithm*.

Approximation algorithms are often associated with NP-hard problems; since it is unlikely that there can ever be efficient polynomial time exact algorithms solving NP-hard problems, one settles for polynomial time sub-optimal solutions.

We present polynomial time approximation algorithms for the minimum vertex and edge guard problems in a polygon P with or without holes.

1. S. K. Ghosh, *Approximation algorithms for art gallery problems*, Technical report no. JHU/EECS-86/15, Department of Electrical Engineering and Computer Science, The Johns Hopkins University, August 1986. Also in Proceedings of the Canadian Information Processing Society Congress, pp. 429-434, 1987. Running time: $O(n^5 \log n)$ time. Approximation ratio: $O(\log n)$.
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3. A. Efrat and S. Har-Peled, *Guarding galleries and terrains*, Information Processing Letters, 100 (2006), 238-245. Running time and the approximation ratio: (i) For simple polygons, $O(nc_{opt}^2 \log^4 n)$ expected time, and $O(\log c_{opt})$ approximation ratio, where c_{opt} is the number of vertices in the optimal solution. (ii) For polygons with h holes, $O(nhc_{opt}^3 polylog\ n)$ expected time, $O(\log n \log(c_{opt} \log n))$ approximation ratio.

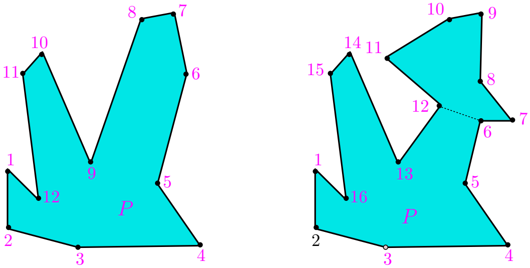
4. B. J. Nilsson, *Approximate Guarding of Monotone and Rectilinear Polygons*, Proceedings of the 32nd International Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science, Springer-Verlag, no. 3580, pp. 1362-1373, 2005. The paper gives polynomial time approximation algorithms (i) for monotone polygons and (ii) for simple orthogonal polygons. Approximation ratios are 12 and 96 respectively.
5. A. Deshpande, T. Kim, E. D. Demaine1 and S. E. Sarma, *A pseudopolynomial time O(log n)-approximation algorithm for art gallery problems* , Proceedings of the 10th International Workshop on Algorithms and Data Structures, LNCS, Springer-Verlag, no. 4619, pp. 163-174, 2007. Running time: Polynomial in n, the number of walls and the spread, where the spread can be exponential. Approximation ratio: $O(\log c_{opt})$.
6. J. King and D. Kirkpatrick, *Improved Approximation for Guarding Simple Galleries from the Perimeter*, Discrete & Computational Geometry, 46(2011) 252-269. Running time: Polynomial in n. Approximation ratio: $O(\log \log c_{opt})$

Heuristics for Stationary Guard Problems

- Recently, efforts are being made to design heuristics to solve stationary guard problems where the efficiency of these heuristics are evaluated by experimentation.
- These heuristics essentially follow Ghosh's method of first discretizing the entire region of a polygon and then using the minimum set-cover solution.
- However, these heuristics use different criteria for discretization or in choosing candidate sets.

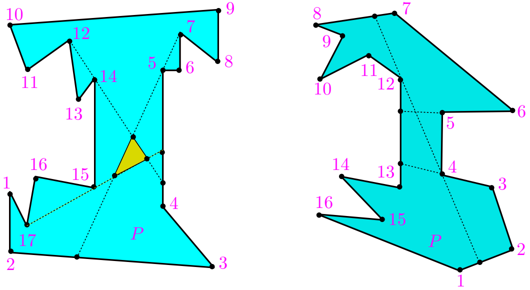
1. Y. Amit and J. S. B. Mitchell and E. Packer, *Locating Guards for Visibility Coverage of Polygons*, Proceedings of the 9th Workshop on Algorithm Engineering and Experiments (ALENEX'07), SIAM, pp. 120-134, 2007.
2. M. C. Couto and P. J. de Rezende and C. C. de Souza, *An exact and efficient algorithm for the orthogonal art gallery problem*, Proceedings of the 20th Brazilian Symposium on Computer Graphics and Image Processing, pp. 87-94, 2007.
3. M. C. Couto and P. J. de Rezende and C. C. de Souza, *Experimental evaluation of an exact algorithm for the orthogonal art gallery problem*, Proceedings of the 7th International Workshop on Experimental Algorithms, LNCS, vol. 5038, pp. 101-113, Springer, Heidelberg, 2008.
4. M. C. Couto and P. J. de Rezende and C. C. de Souza, An IP solution to the art gallery problem, Proceedings of the 25th Annual ACM Symposium on Computational Geometry, pp. 88-89, 2009.

Vertex-guard problem



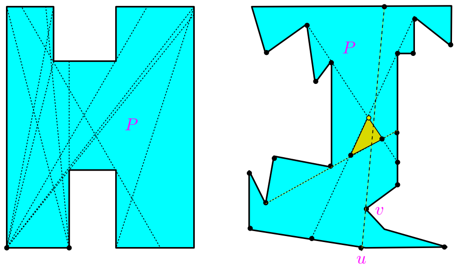
A simple polygon is called a *fan* if there exists a vertex that is visible from all points in the interior of the polygon.

The vertex guard problem can be treated as a polygon decomposition problem in which the decomposition pieces are fans.



Vertices 7, 12 and 17 together can see the entire boundary of the polygon but the shaded region is not visible from any of these vertices.

Three fans (vertices 1, 4 and 7) are necessary to cover the polygon if only edge extensions are allowed, whereas two fans (vertices 1 and 7) suffice if we allow the boundary of convex components to be bounded by segments that contains any two vertices of the polygon.

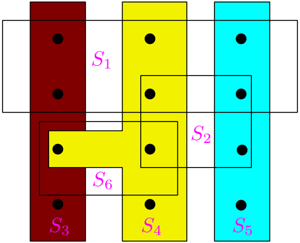


The polygonal region of P is decomposed into convex components where every component is bounded by segments that contains any two vertices of the polygon.

Every convex component must lie in at least one of the fans chosen by the approximation algorithm.

Lemma: Every convex component is either totally visible or totally invisible from a vertex of P.

Minimum set-covering problem



Minimum set-covering problem: Given a finite family C of sets $S_1, \ldots, S_n$, the problem is to determine the minimum cardinality $A \subseteq C$ such that $\bigcup_{i \in A} S_i = \bigcup_{j=1}^{n} S_j$.

The problem of finding the minimum number of fans to cover P is same as the minimum set-covering problem, where every fan is a set and convex components are elements of the set.

1. D. S. Johnson, *Approximation algorithms for combinatorial problems*, Journal of Computer and System Sciences, 9 (1974), 256-278.
2. V. Vazirani, *Approximation algorithms*, Springer, 2003.

Vertex-guard algorithm

Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.
Step 2: *For* $1 \leq j \leq n$, construct the set F_j by adding those convex components of P that are totally visible from the vertex v_j.
Step 3: Find $i \in N$ such that $|F_i| \geq |F_j|$ *for all* $j \in N$ and $i \neq j$.
Step 4: Add i to Q and delete i from N.
Step 5: *For all* $j \in N$, $F_j := F_j - F_i$, and $C := C - F_i$.
Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.
Step 7: Output the set Q and Stop.

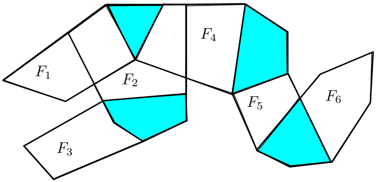
Theorem: The approximation algorithm for the minimum vertex guard problem in a polygon P of n vertices computes solutions that are at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

Edge-guard algorithm

Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.
Step 2: *For* $1 \leq j \leq n$, construct the set E_j by adding those convex components of P that are totally visible from the edge e_j of P.
Step 3: Find $i \in N$ such that $|E_i| \geq |E_j|$ *for all* $j \in N$ and $i \neq j$.
Step 4: Add i to Q and delete i from N.
Step 5: *For all* $j \in N$, $E_j := E_j - E_i$, and $C := C - E_i$.
Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.
Step 7: Output the set Q and Stop.
Theorem: For the minimum edge guard problem in an n-sided polygon P, an approximate solution can be computed which is at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

Conjecture

Any set consisting of arbitrary chosen convex components may not form a fan as every fan consists of contiguous convex components. Therefore, constructing any example where the greedy algorithm takes $O(\log n)$ times optimal does not seem to be possible.



Conjecture: (Ghosh 1986) Approximation algorithms are expected to yield solutions within a constant factor of the optimal.

Lower bound on approximation ratio

Regarding the lower bound on the approximation ratio for the problems of minimum vertex, point and edge guards in simple polygons, it has been shown that these problems are APX-hard using gap-preserving reductions from 5-OCCURRENCE-MAX-3-SAT.

This means that for each of these problems, there exists a constant $\epsilon > 0$ such that an approximation ratio of $1 + \epsilon$ cannot be guaranteed by any polynomial time approximation algorithm unless $P = NP$.

The above statement implies that there may be approximation algorithms for these problems whose approximation ratios are not small constants.

On the other hand, for polygons with holes, these problems cannot be approximated by a polynomial time algorithm with ratio $((1-\epsilon)/12)(ln\ n)$ for any $(\epsilon > 0)$, unless $NP \subseteq TIME(n^{O(loglogn)})$. The results are obtained by using gap-preserving reductions from the SET COVER problem.

Ghosh's $O(\log n)$-approximation algorithms are optimal for polygons with holes.

Open problems

Design approximation algorithms for vertex, edge and point guards problems in simple polygons which yield solutions within a constant factor of the optimal.

1. S. Eidenbenz C. Stamm and P. Widmayer, *Inapproximability Results for Guarding Polygons and Terrains*, Algorithmica, 31 (2000), 79-113.
2. P. Bhattachary, S. K. Ghosh and B. Roy, *Vertex Guarding in Weak Visibility Polygons*, Proceedings of the 1st International Conference on Algorithms and Discrete Applied Mathematics, Kanpur, Lecture Notes in Computer Science, vol. 8959, pp. 45-57, Springer, 2015.

An $O(1)$-approximation algorithm

We present a 6-approximation algorithm, which has running time $O(n^2)$, for vertex guarding polygons that are weakly visible from an edge uv and contain no holes.

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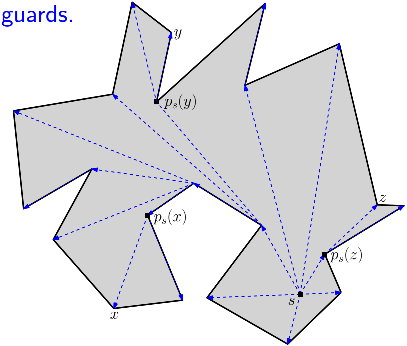
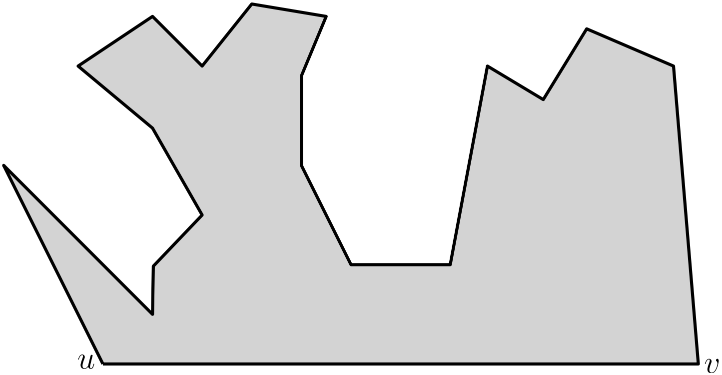
Our algorithm uses Euclidean shortest path trees from u and v for choosing vertices for placing guards.



Euclidean shortest path tree rooted at s. The parents of vertices x, y and z in $SPT(s)$ are marked as $p_s(x)$, $p_s(y)$ and $p_s(z)$ respectively.

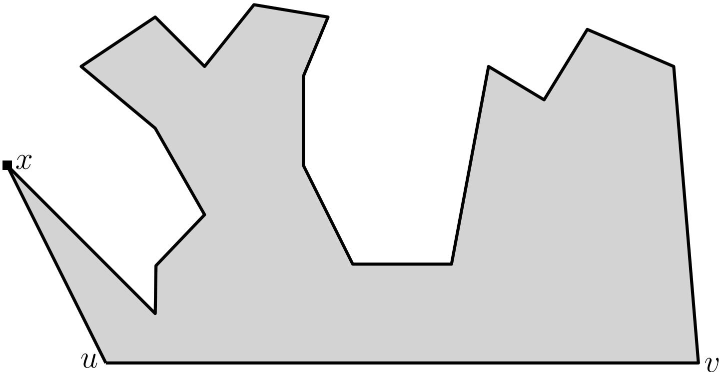
1. P. Bhattachary, S. K. Ghosh and B. Roy, *Vertex Guarding in Weak Visibility Polygons*, Proceedings of the 1st International Conference on Algorithms and Discrete Applied Mathematics, Kanpur, Lecture Notes in Computer Science, vol. 8959, pp. 45-57, Springer, 2015.

A naive algorithm for guarding all vertices



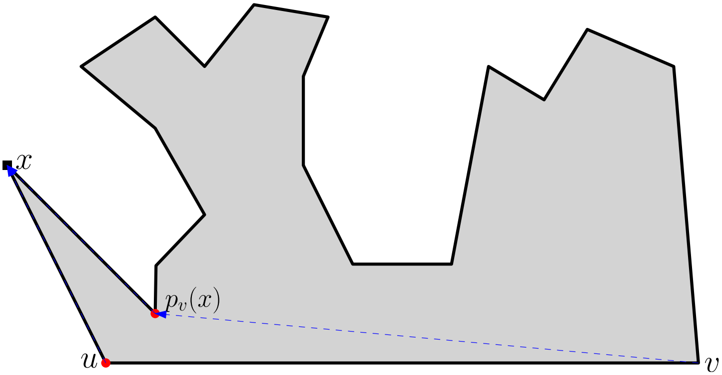
$A = \{\}; S = \{\}$

A naive algorithm for guarding all vertices



$A = \{x\}$; $S = \{\}$

A naive algorithm for guarding all vertices



$A = \{x\}$; $S = \{u, p_v(x)\}$

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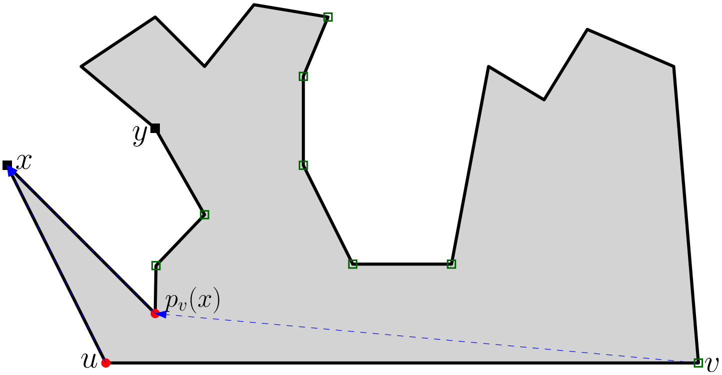

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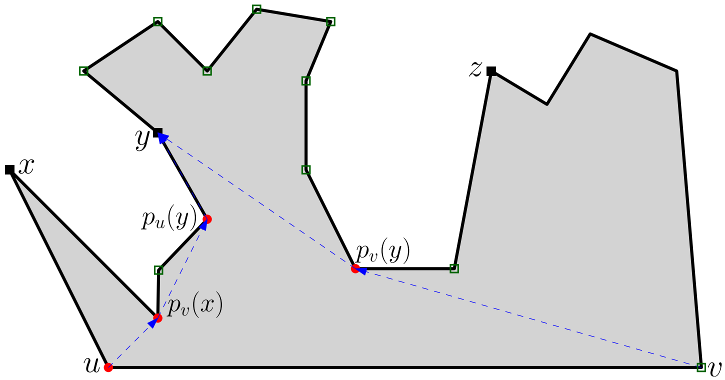
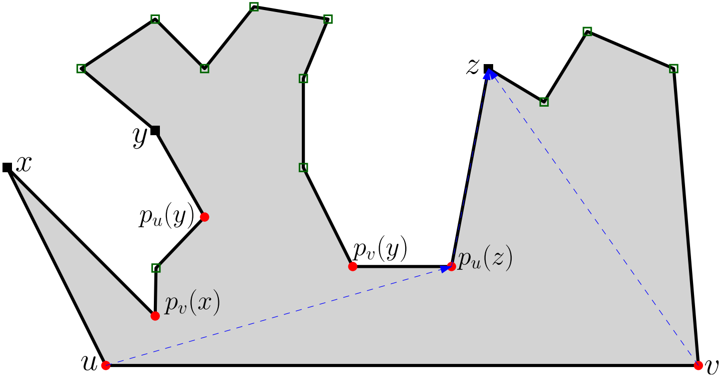


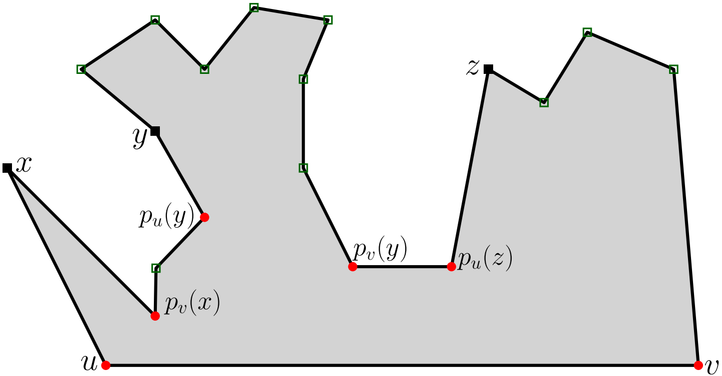
$A = \{x, y\}$; $S = \{u, p_v(x), p_u(y), p_v(y)\}$

A naive algorithm for guarding all vertices



$A = \{x, y, z\}$; $S = \{u, p_v(x), p_u(y), p_v(y), p_u(z), v\}$

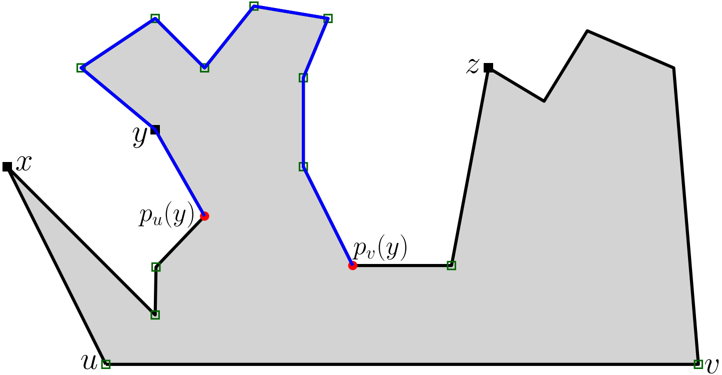
A naive algorithm for guarding all vertices



$A = \{x, y, z\}$; $S = \{u, p_v(x), p_u(y), p_v(y), p_u(z), v\}$

Observe that $|S| = 2|A|$

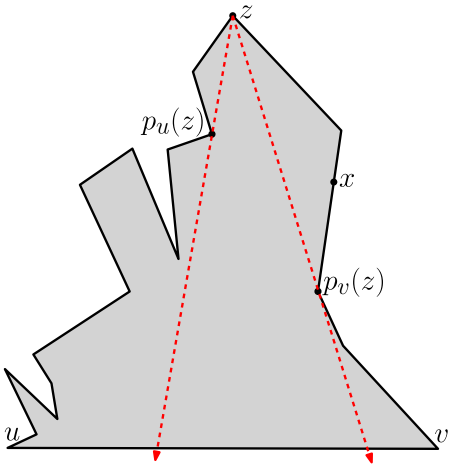
Performance guarantee under a special condition



Observe that the vertex $y \in A$ is such that every vertex lying on the clockwise boundary between $p_u(y)$ and $p_v(y)$ [henceforth denoted as $bd_c(p_u(y), p_v(y))$] is visible from $p_u(y)$ or $p_v(y)$.

If each vertex $z \in A$ is such that every vertex of $bd_c(p_u(z), p_v(z))$ is visible from $p_u(z)$ or $p_v(z)$, then $|S| \leq 2|S_{opt}|$ as follows.

Location of an optimal guard for vertex z



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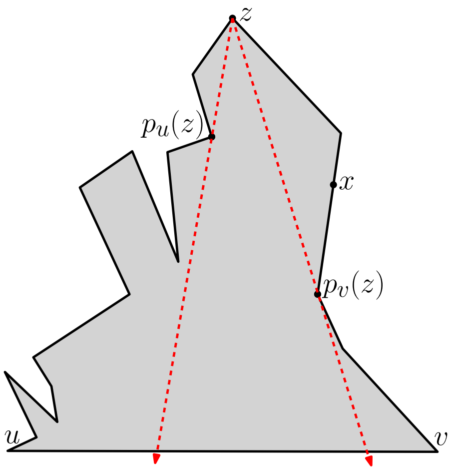
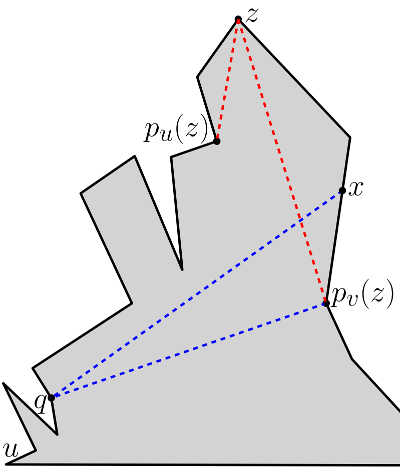
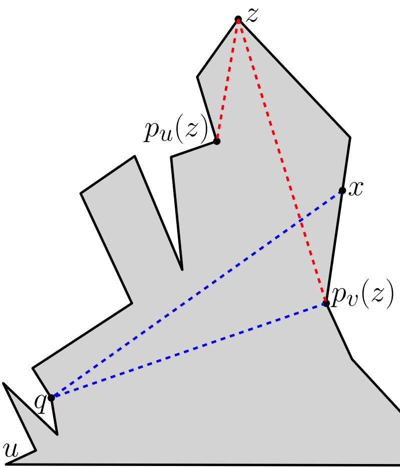


Any guard $x \in S_{opt}$ that sees z must lie on $bd_c(p_u(z), p_v(z))$.



- All vertices of $bd_c(p_u(z), p_v(z))$ are visible from $p_u(z)$ or $p_v(z)$.



- All vertices of $bd_c(p_u(z), p_v(z))$ are visible from $p_u(z)$ or $p_v(z)$.
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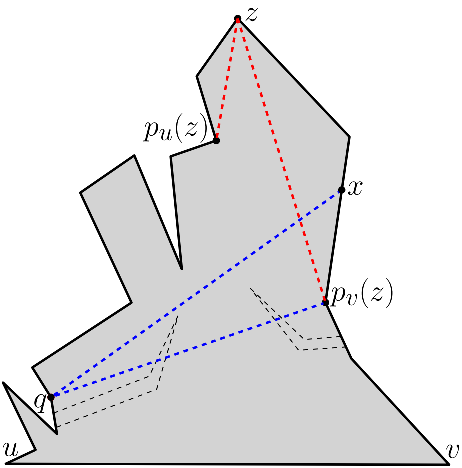

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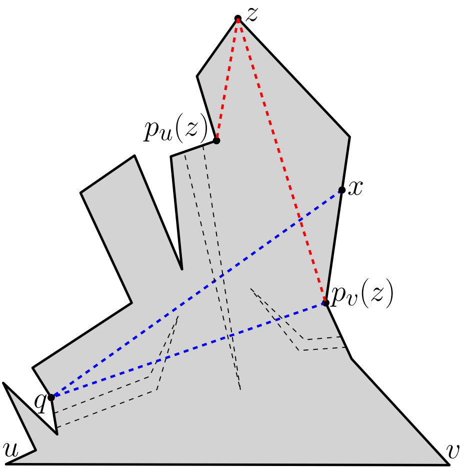
- If any point q is visible from x, then q must be visible from $p_u(z)$ or $p_v(z)$.



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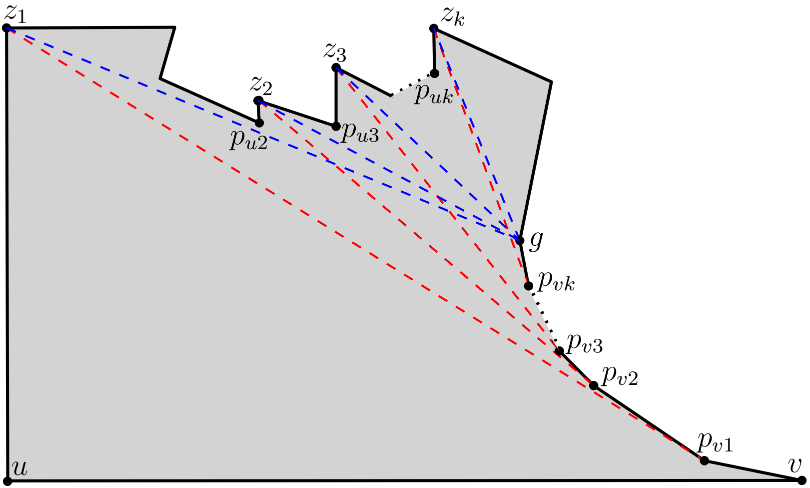
- If q is visible from x, then q must be visible from $p_u(z)$ or $p_v(z)$.
- This implies that $|A| \leq |S_{opt}|$.

A bad input polygon for the naive algorithm



For this input instance, $|S| = 2k$, whereas $S_{opt} = \{u, g\}$.

A 4-approximation algorithm for guarding all vertices

For the current unmarked vertex z, an invariance is maintained such that every vertex of $bd_c(u, z)$ is visible from some guard in $S \cup \{p_u(z), p_v(z)\}$.

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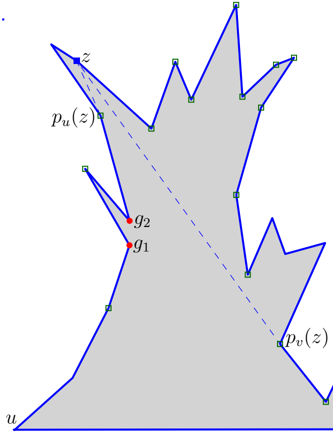

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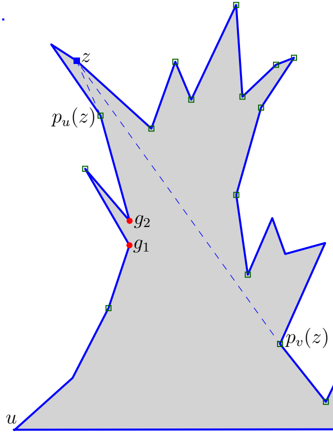

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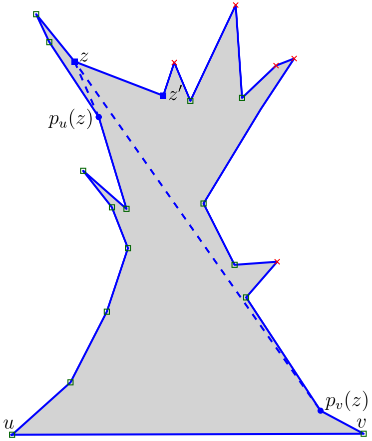
Case 2: There exist some vertices lying on $bd_c(z, p_v(z))$, not visible already from guards currently in S, such that they do not become visible even if new guards are placed at $p_u(z)$ and $p_v(z)$.



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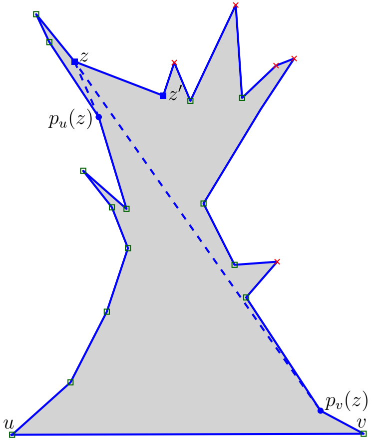
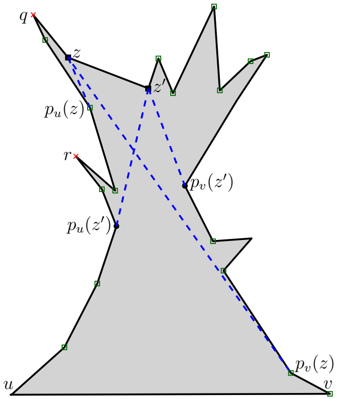


Let z' be the next vertex along the clockwise scan that is not visible from any guard already in S.

Case 2a: Not every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from $p_u(z')$ or $p_v(z')$. (Invariance does not hold for z'.)



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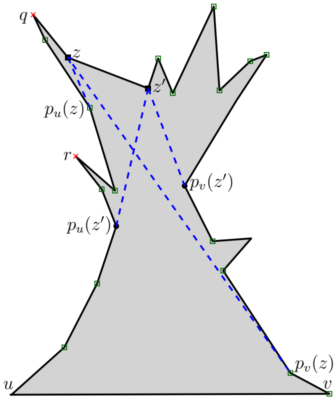

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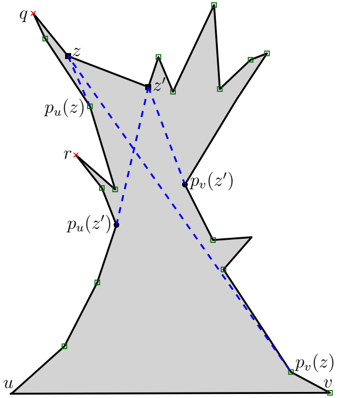

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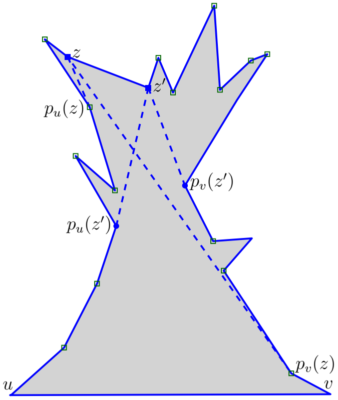
After these operations, invariance holds for z'.

Case 2b: Every unmarked vertex of $bd_c(p_u(z'), z')$ is visible from $p_u(z')$ or $p_v(z')$. [The vertex z is skipped as z' satisfies the invariance.]



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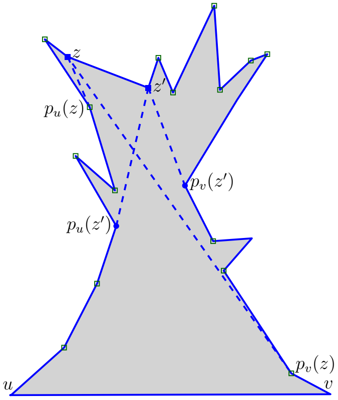

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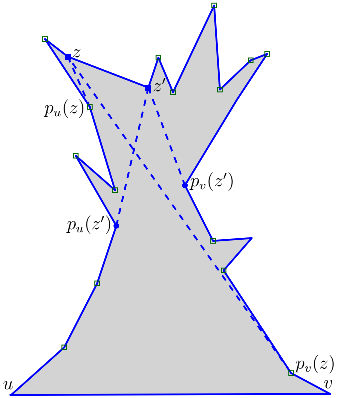

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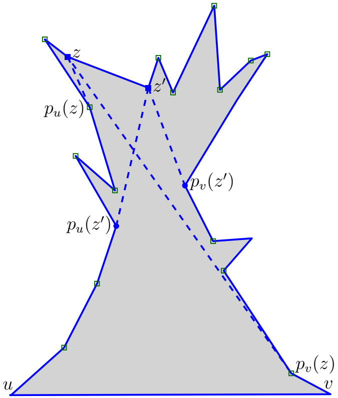

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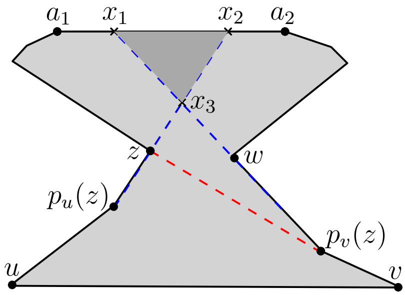
Therefore, $|B| \leq 2|S_{opt}|$ and hence, $|S| = 2|B| \leq 4|S_{opt}|$.

Insufficiency of guards in S to cover all interior points



All vertices are visible from the guard set $S = \{p_u(z), p_v(z)\}$, but all points in the triangular interior region $x_1x_2x_3$ are not visible.

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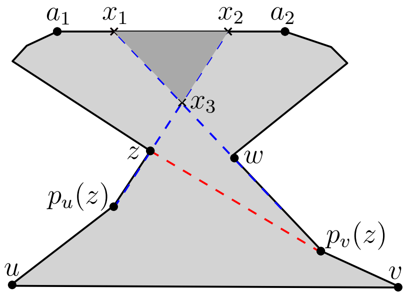

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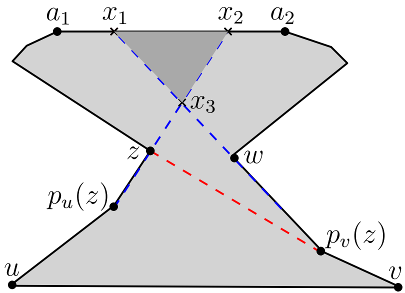

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In fact, for any such invisible region, one of the sides must always be part of a polygonal edge.

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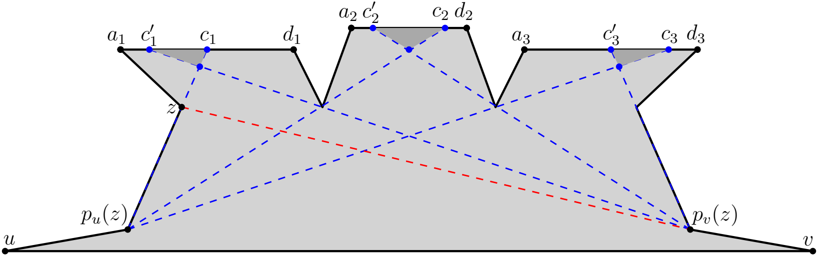
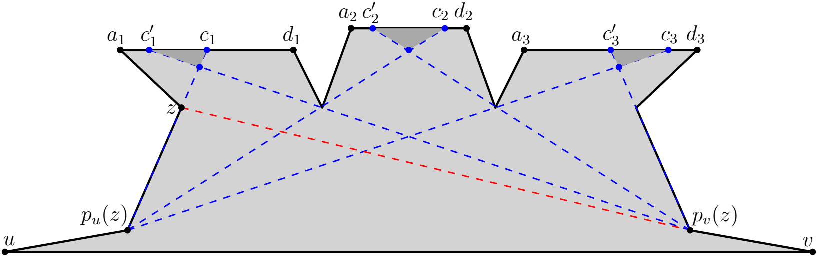


Multiple invisible regions exist within the polygon that are not visible from the guard set $S = \{p_u(z), p_v(z)\}$.

Placement of more guards to cover all interior points



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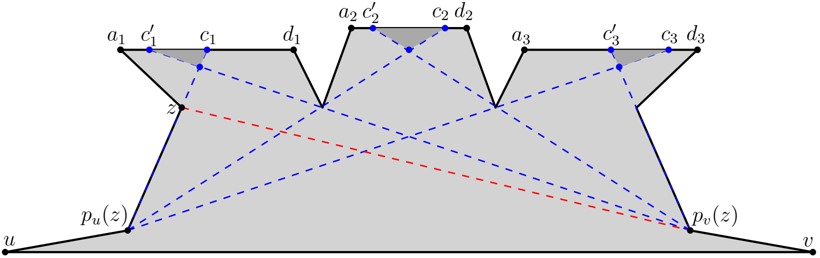

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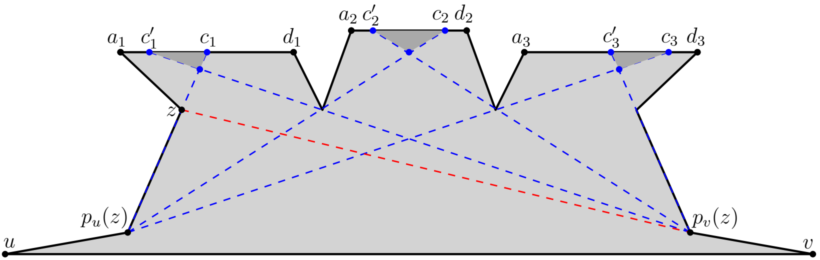

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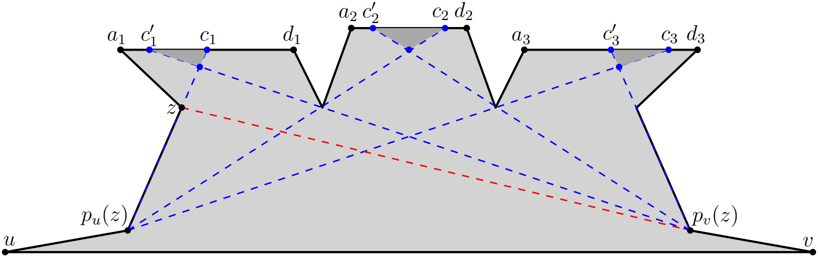

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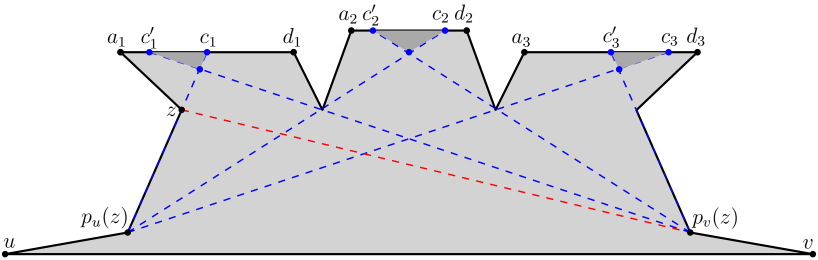

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Theorem: The approximation algorithm for the minimum vertex guard problem in an n-sided weak visibility polygon runs in $O(n^2)$ time and computes a solution that is at most six times the optimal.

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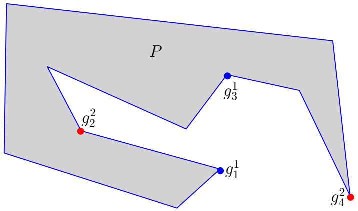

Two colors and four guards are sufficient for this polygon.

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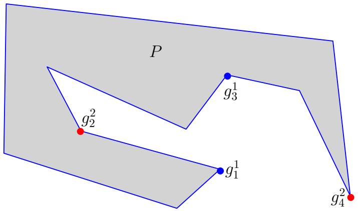

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Such placement of color guards in P is called *strong conflict-free coloring of P*.

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Such placement of color guards in P, where every point $z \in P$ sees one guard whose color is different from all other guards visible from z, is called *weak conflict-free coloring of* P.

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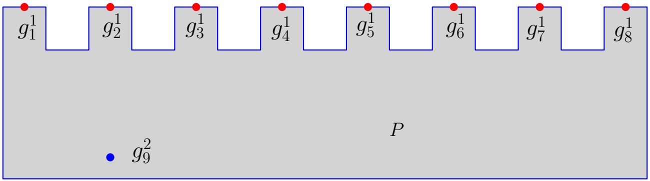

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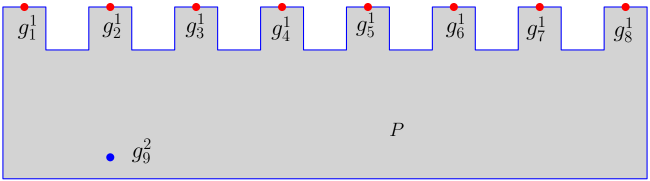
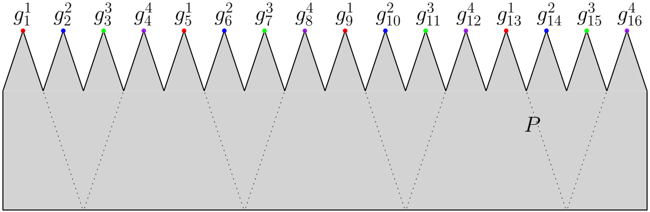
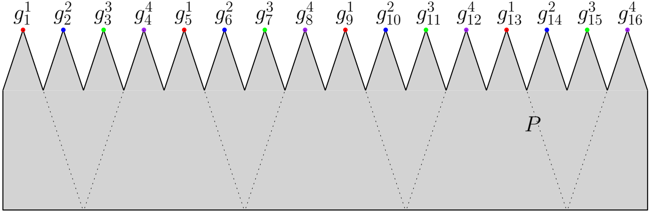


Two colors for guards are sufficient for weak conflict-free coloring of P.

Observe that any strong conflict-free coloring is also a weak conflict-free coloring.



For a strong conflict-free coloring, a monotone polygon P requires $\Omega(\sqrt{n})$.



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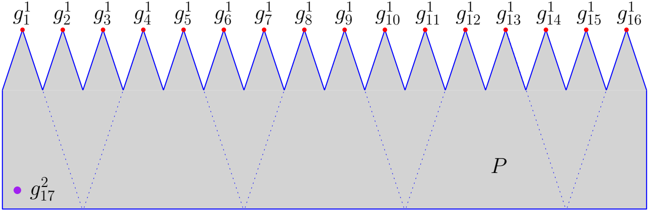


For a weak conflict-free coloring, the same polygon requires 2 colors.

1. L. H. Erickson and S. M. LaValle, *An art gallery approach to ensuring that landmarks are distinguishable*, Robotics: Science and Systems, 2011.
2. Andreas Bärtschi and Subhash Suri, *Conflict-free chromatic art gallery coverage*, Algorithmica, vol. 68, pp. 265-283, 2014.

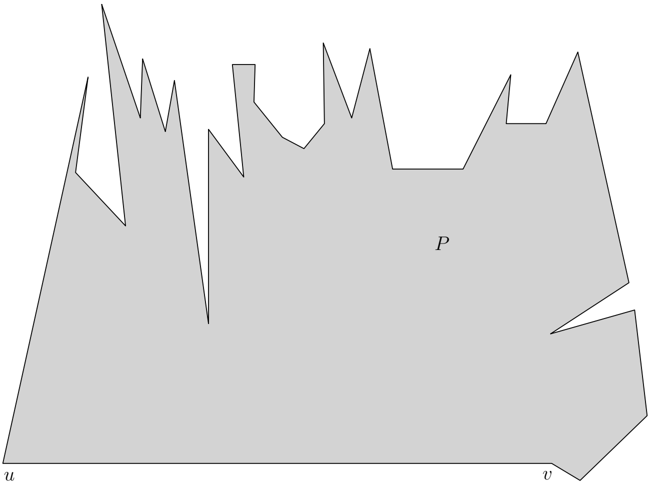
Best known bounds on chromatic guard number

Simple polygon	Strong conflict-free chromatic number	Weak conflict-free chromatic number
Upper bound	$\lfloor n/3 \rfloor$ from art gallery theorem	$O(\log n)$ by Bartschi et al.
lower bound	$\lfloor n/4 \rfloor$ by Erickson et al.	$O((\log \log n)/(\log \log \log n))$ by Hoffmann et al.

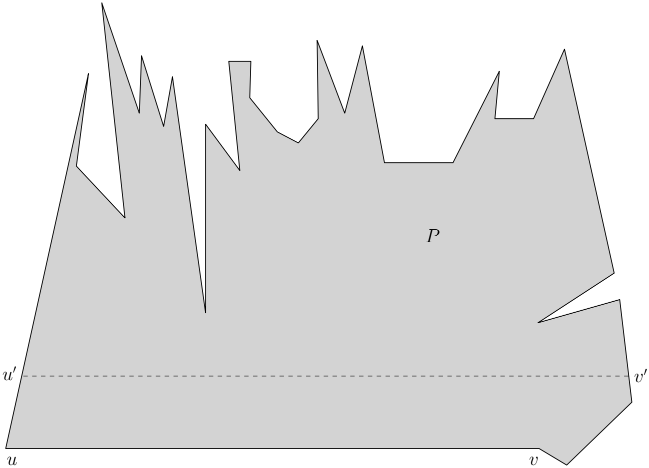
Polygon with holes	Strong conflict-free chromatic number	Weak conflict-free chromatic number
Upper bound	$\left\lfloor \frac{n+h}{3} \right\rfloor$	$\left\lfloor \frac{n+h}{3} \right\rfloor$
lower bound	$\Omega(1)$	$\Omega(1)$

1. A. Baertschi, S. K. Ghosh, M. Mihalak, T. Tschager and P. Widmayer, *Improved bounds for the conflict-free chromatic art gallery problem*, Proceedings of the 30th ACM Annual Symposium on Computational Geometry, pp. 144-153, 2014.
2. F. Hoffmann, K. Kriegel and M. Willert, *Almost tight bounds for conflict-free chromatic guarding of orthogonal galleries*, CoRR abs/1412.3984, 2014.

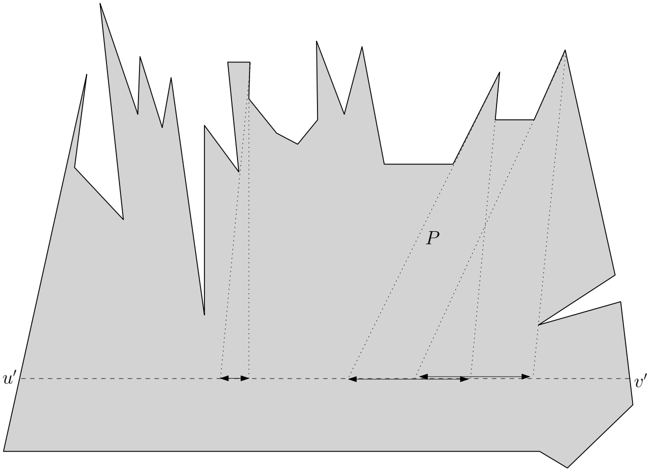
Conflict-free coloring of a weak visibility polygon



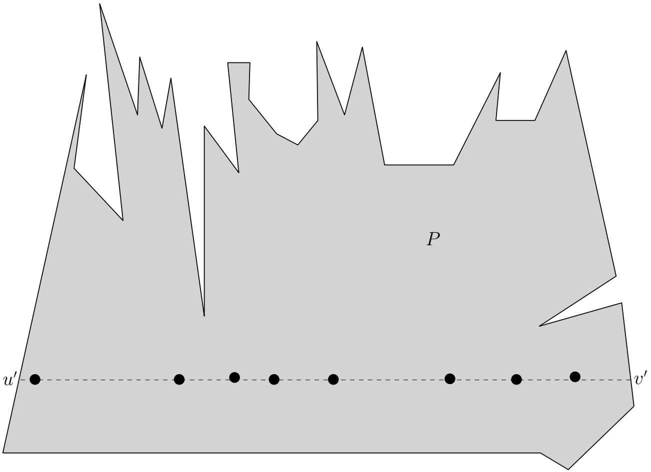
P is weakly visible from the edge uv.



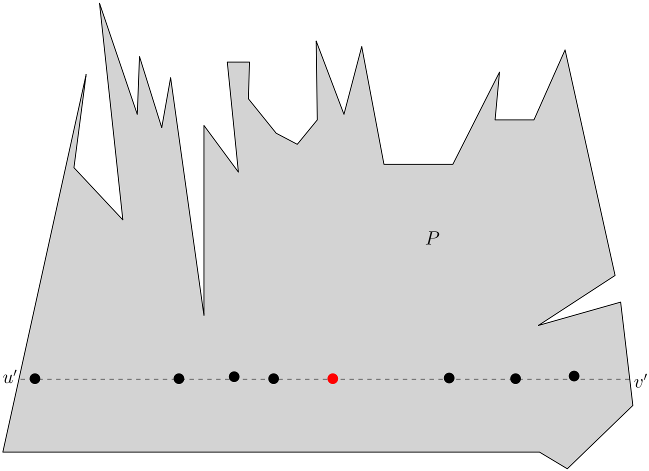
Draw a line segment $u'v'$ parallel to uv in P, where $u'v'$ is slightly above uv.



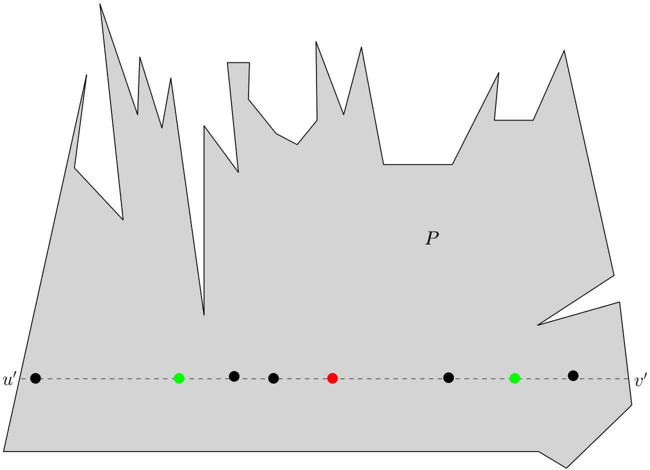
Every vertex of P is visible from points of an interval on $u'v'$.



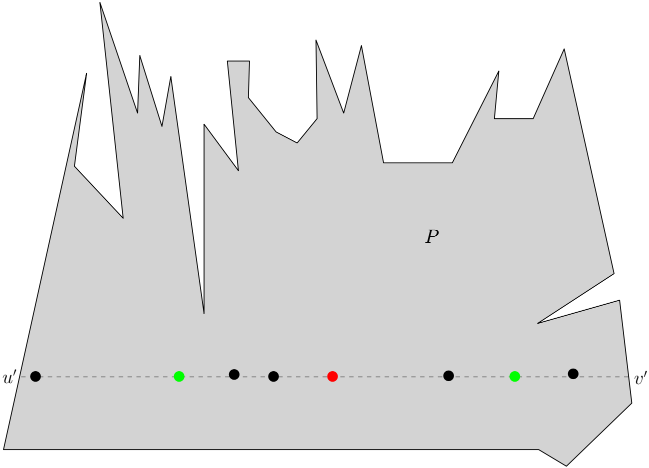
By scanning interval from the left to right, place guards on $u'v'$ such that the entire P is visible from these guards.



Take the middle guard among the guards on $u'v'$ and color it with red.

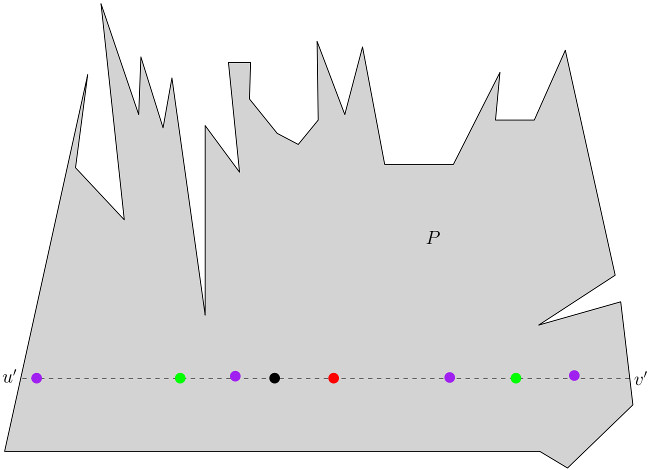


Take the middle guard among the guards on the left of red guard, and color it with green.

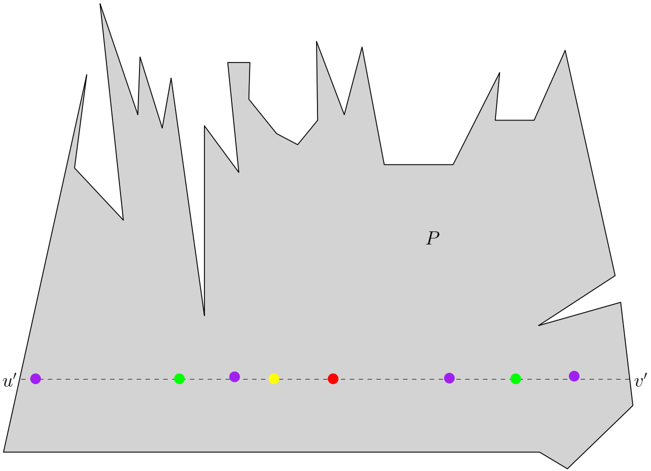


Take the middle guard among the guards on the left of red guard, and color it with green.

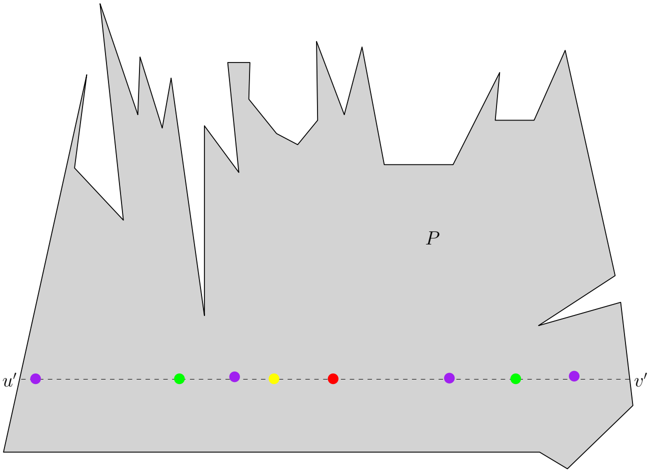
Similarly, take the middle guard among the guards on the right of red guard, and color it with green.



The process of coloring continues recursively.



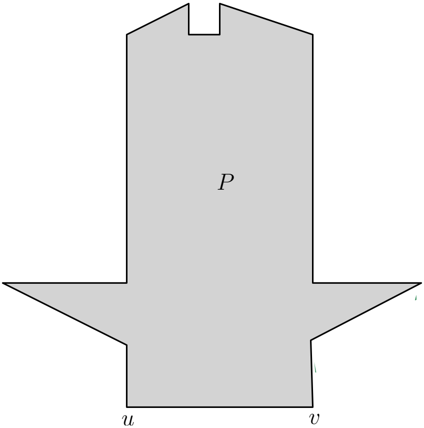
Any point z of P sees an unique colored guard.



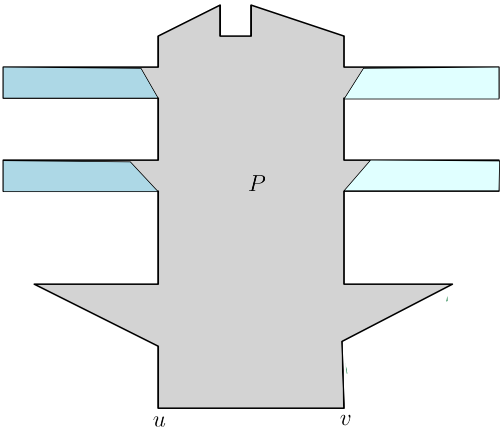
Any point z of P sees an unique colored guard.

Theorem: A weak conflict-free coloring of a weak visibility polygon P can be done with $O(\log n)$ colors.

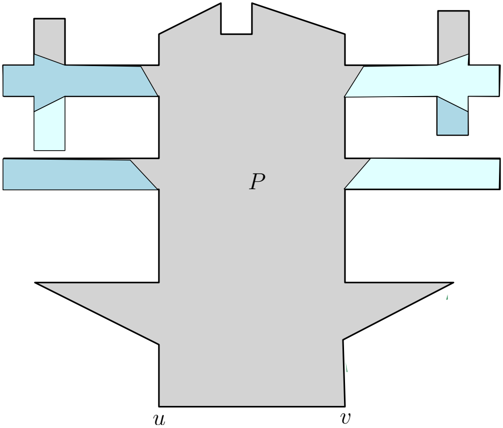
Conflict-free coloring of a simple polygon



Since P is weakly visible from uv, one set of $O(\log n)$ colors are enough.



P is not weakly visible from uv.
(i) One additional set of $O(\log n)$ color is required for left pockets.
(ii) Another set of $O(\log n)$ color is required for right pockets.



Theorem: A weak conflict-free coloring of a simple polygon P can be done with three sets of $O(\log n)$ colors.

Concluding remarks

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Thank you very much indeed.