Geometric Spanners

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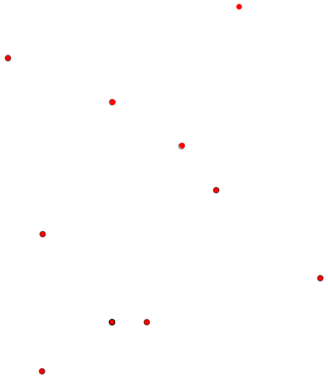
Outline

Designing a Network

How do we connect these cities?



Connect everybody!

Designing a Network

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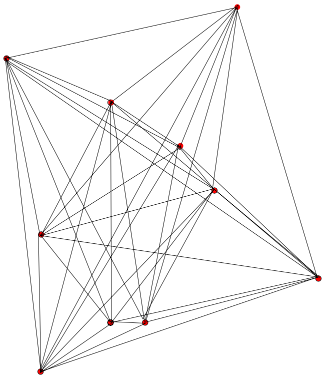

Too many edges!

Designing a Network

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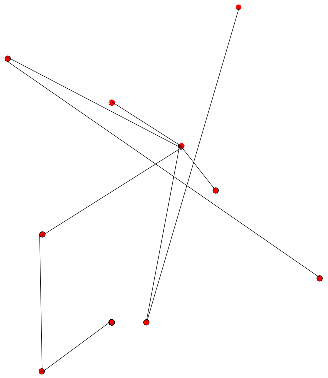

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Network

Network (Graph)

Edge-weighted undirected graph $G(V, E)$
V= Set of Vertices
E=Set of Edges

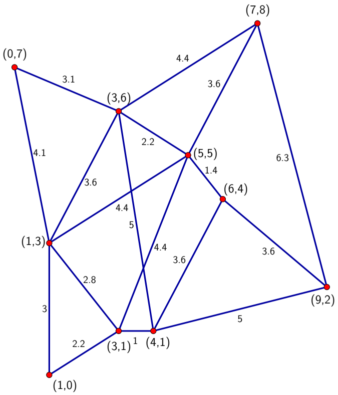
- $\forall e = (u, v) \in E$,

$$wt(e) = \mathbf{d}(u, v).$$

Geometric Network

- V= Set of points in the plane.
- $\mathbf{d}$ = Euclidean distance, i.e., $\forall e = (u, v) \in E$, $wt(e) = |uv|$.

Network



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Geometric Network

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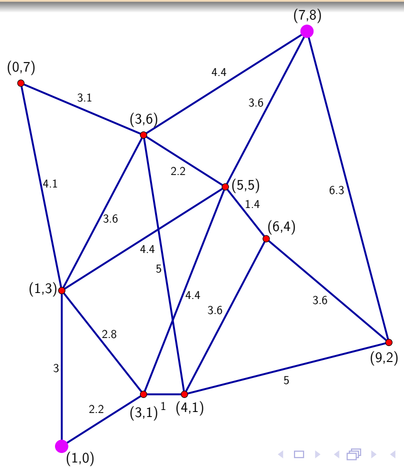
t-spanner

Network $G(V, E)$ is a t-spanner ($t \geq 1$) if

- $\forall u, v \in V$,

$$\mathbf{d}_G(u, v) \leq t \times \mathbf{d}(u, v). \qquad (t\text{-path})$$



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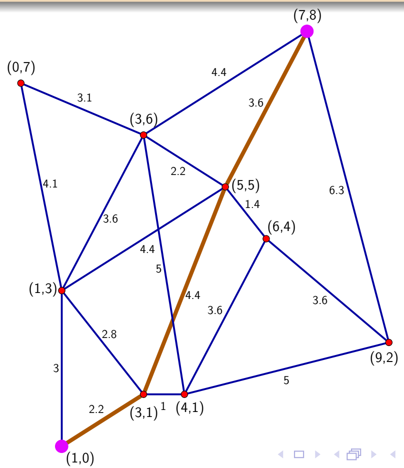

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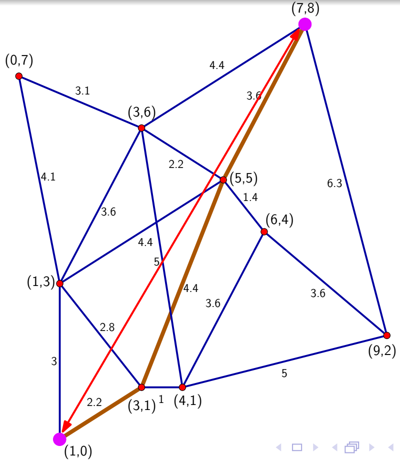
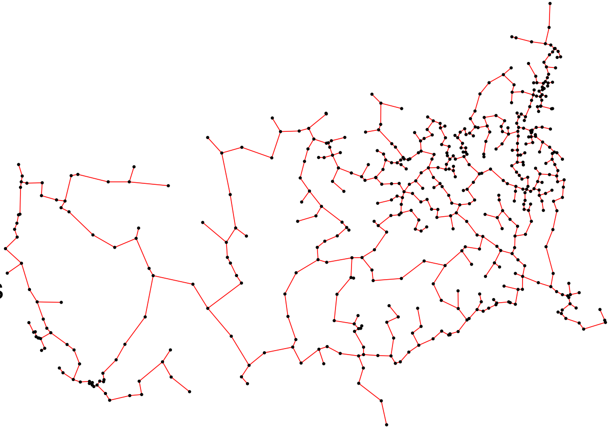


Example

10-spanner

for

532 US-cities

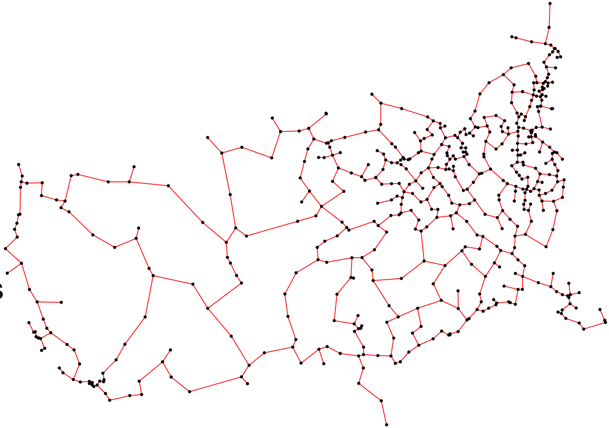


Example

5-spanner

for

532 US-cities

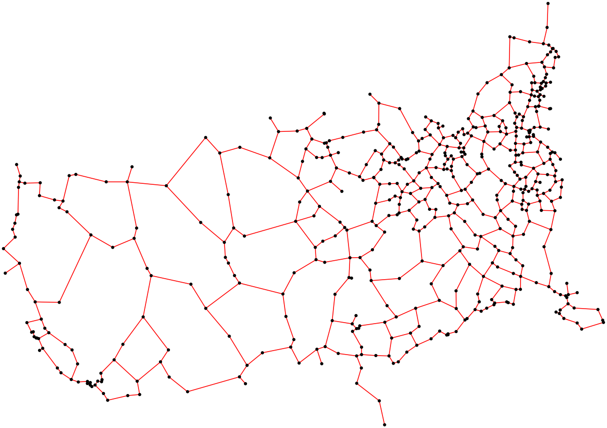


Example

3-spanner

for

532 US-cities

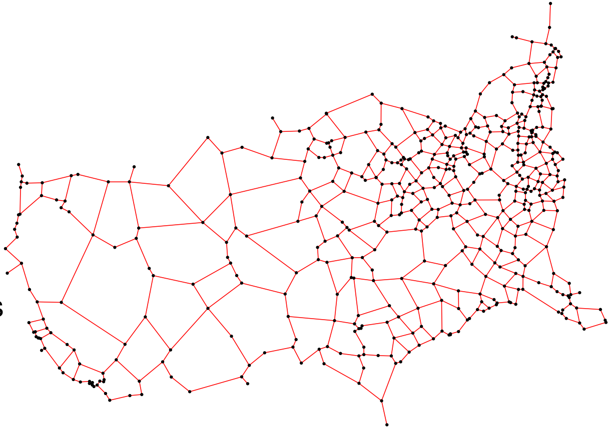


Example

2-spanner

for

532 US-cities

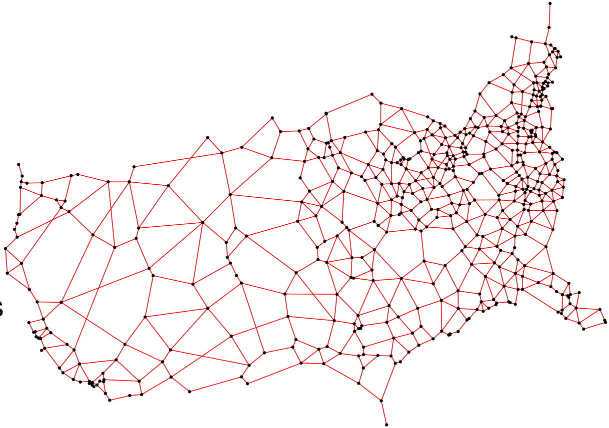


Example

1.5-spanner

for

532 US-cities



Main Question

Problem:

Given a set V and $t > 1$, construct a sparse t-spanner of V.

Sparseness:

- Number of edges (size)
- Weight (compared with weight of MST)
- Maximum degree
- Diameter

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Spanner Algorithms

Spanner Algorithms

-	Size	Weight / $wt(MST)$	Degree	Time	
Greedy	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n^3 \log n)$	Metric
Apx. greedy	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n \log n)$	Geometric
Θ-graph	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n \log n)$	Geometric
WSPD	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$	$\mathcal{O}(n \log n)$	Geometric
Sink-spanner	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(n \log n)$	Geometric
Skip-list spanner	$\mathcal{O}(n)^*$	$\mathcal{O}(n)^*$	$\mathcal{O}(n)$	$\mathcal{O}(n \log n)^*$	Geometric

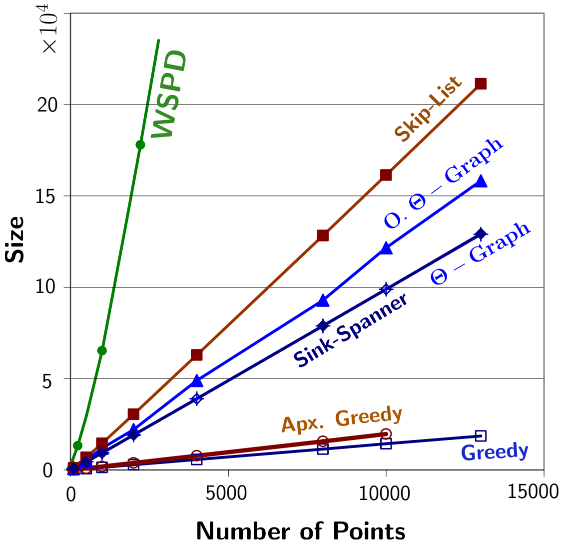
Constructing sparse t-spanners:

- Greedy (Bern (1989) and Althöfer et al. (1993)).
- Θ-graph (Clarkson (1987) and Keil (1988)).
- Well-Separated Pair Decomp. (Callahan and Kosaraju (1992)).

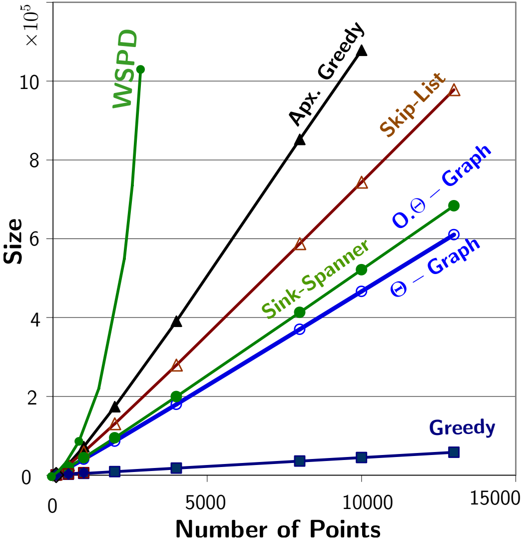
Spanner Algorithms

In Practice

2-spanner



1.1-spanner



Size
$\times 10^5$
0
2
4
6
8
10
0
5000
10000
15000
Number of Points
WSPD
Apx. Greedy
Skip-List
O.Θ – Graph
Sink-Spanner
Θ – Graph
Greedy

Greedy Algorithm

- Works for any metric, but this talk is only about points in the plane and Euclidean distances.
- Sort all pairs of points in increasing order of distances.
- Consider them one by one in that order.
- Let (u, v) be the current pair under consideration.
- Add e to the spanner if the distance between the endpoints u and v is greater than $t \cdot \mathrm{d}(u, v)$.

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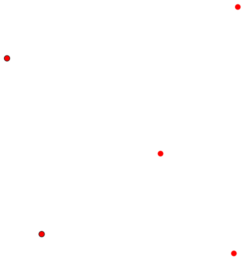
An Example ($t = 2$)



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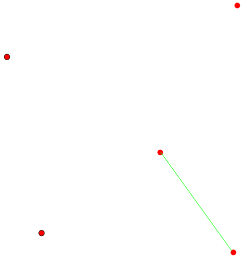

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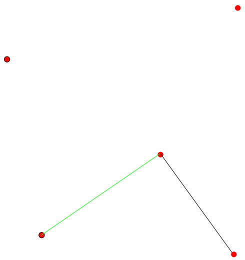

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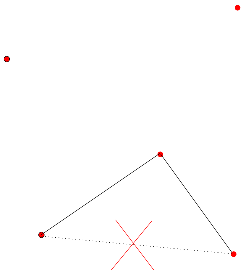

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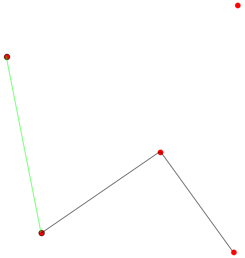

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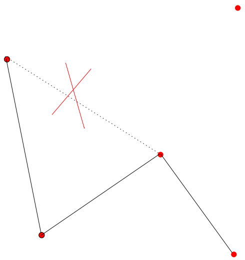

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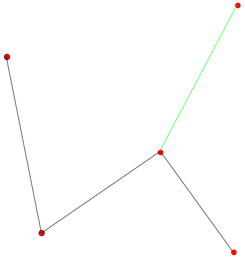

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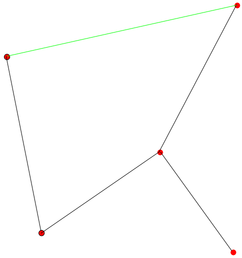

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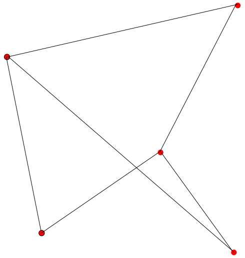

Original Greedy Algorithm

$E := \emptyset; G := (V, E);$

foreach $(u, v) \in V^2$ *(in sorted order)* **do**

  if $\mathbf{d}_G(u, v) > t \cdot \mathbf{d}(u, v)$ **then**

    $E := E \cup \{(u, v)\};$

  end

end

return $G = (V, E);$

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Running time:$\mathcal{O}(n^2 \times SSSP) = \mathcal{O}(mn^2 + n^3 \log n)$.
(Dijkstra's SSSP requires $\mathcal{O}(m + n \log n)$ time)

Properties of Greedy Spanner

- Linear size, i.e., number of edges is $\mathcal{O}(n)$.
- Constant degree.
- Weight is $\mathcal{O}(\log n.\text{wt(MST)})$.
- But running time is VERY HIGH, i.e., $\mathcal{O}(n^2 \times SSSP)$.

Challenge

Can we somehow avoid $\mathcal{O}(n^2)$ shortest path computations?

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FG-Greedy Algorithm

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$E := \emptyset$; $G := (V, E)$;

foreach $(u, v) \in V^2$ *with* $u \neq v$ **do** $weight(u, v) := \infty$;

foreach $(u, v) \in V^2$ *(in sorted order)* **do**

  if $weight(u, v) > t \cdot \mathbf{d}(u, v)$ **then**

    perform an SSSP in G with source u;

    foreach $w \in V$ **do** update $weight(u, w)$ and $weight(w, u)$;

    if $weight(u, v) > t \cdot \mathbf{d}(u, v)$ **then** $E := E \cup \{(u, v)\}$;

  end

end

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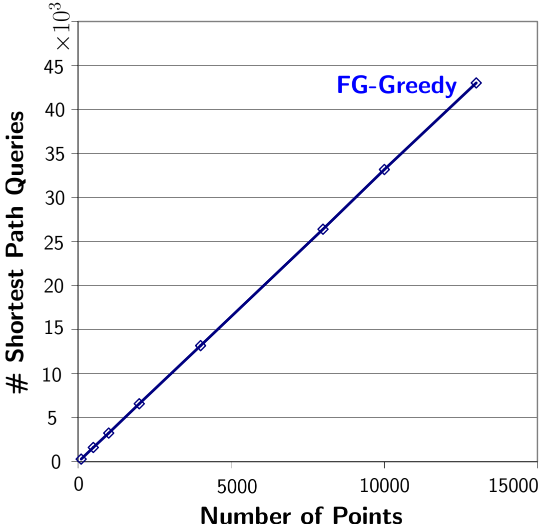
FG-Greedy Algorithm

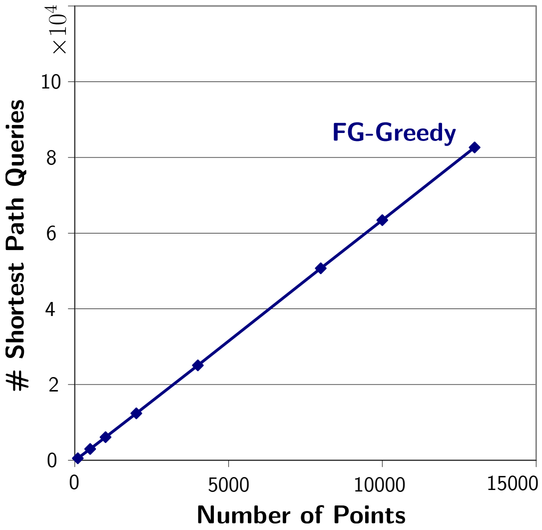
In Practice

$t = 2$



FG-Greedy Algorithm
In Practice

$t = 1.1$



FG-Greedy Algorithm

Counterexample

Farshi-Gudmundsson Conjecture:

The FG-greedy algorithm performs $\mathcal{O}(n)$ SSSP.

FALSE!

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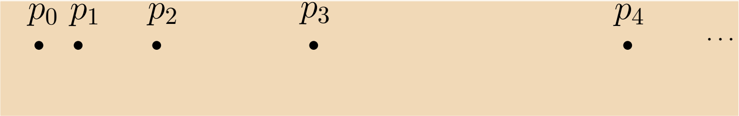
$p_i = 2^i$



Improved Algorithm(2008)

Main ideas:

- Splitting all pairs of points into $\mathcal{O}(n)$ buckets and processing a bucket at a time.
- Keeping the weight matrix up-to-date for all pairs in the processing bucket.
  - Update the weight matrix at the beginning of processing a bucket (by running bounded Dijkstra's algorithm).
  - After adding an edge, update the weight matrix (partially).
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Improved Algorithm

New-Greedy Algorithm

$E := \emptyset$; $G := (V, E)$;

foreach $(u, v) \in V^2$ *with* $u \neq v$ **do** $weight(u, v) := \infty$;

Split all pair of points to buckets $\{E_i\}$;

foreach i **do**

    $\forall$ vertices, run bounded-Dijkstra and update $weight$;

    foreach $(u, v) \in E_i$ *(in sorted order)* **do**

        if $weight(u, v) > t \cdot \mathbf{d}(u, v)$ **then**

            $E := E \cup \{(u, v)\}$;

            Partially update $weight$;

        end

    end

end

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Buckets:

- L_1: distance between the closest pair.
- E_1: all pairs with distance in $[L_1, 2L_1)$.

$\vdots$

- L_i: distance between the closest pair between remaining pairs.
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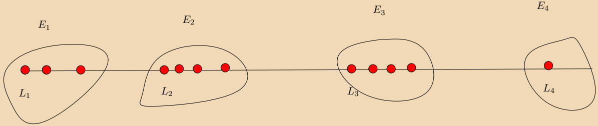

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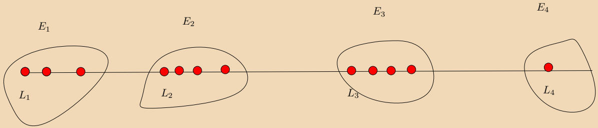

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Theorem: (Har-Peled)

For any set of n points from a metric space, the number of buckets is $\mathcal{O}(n)$.

Algorithm

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- Splitting all pairs of points into $\mathcal{O}(n)$ buckets and processing a bucket at a time.
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Improved Algorithm

Bounded Dijkstra's algorithm

- Assume we are about to process bucket E_i.
- We know that $\forall(u,v) \in E_i, \mathbf{d}(u,v) \in [L_i, 2L_i)$.

Claim:

Running Dijkstra's algorithm with bound $2tL_i$ is sufficient.

Note:

To prove this, we need to show that, at the moment the algorithm process $(u,v) \in E_i$, we have $weight(u,v) > t \cdot d_G(u,v)$ if and only if $d_G(u,v) > t \cdot d(u,v)$.

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The Dijkstra's algorithm stops as soon as the minimum key in the priority queue is larger than $2tL_i$.

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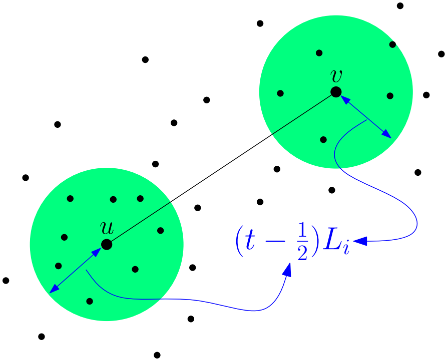
Algorithm

Updating the weight matrix after adding an edge



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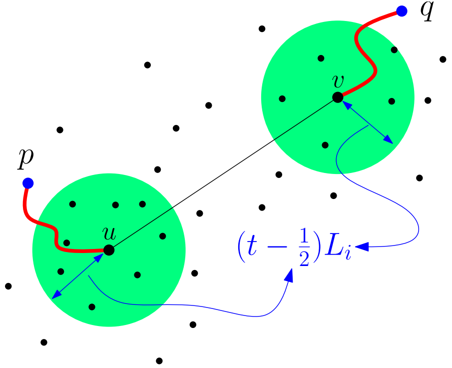

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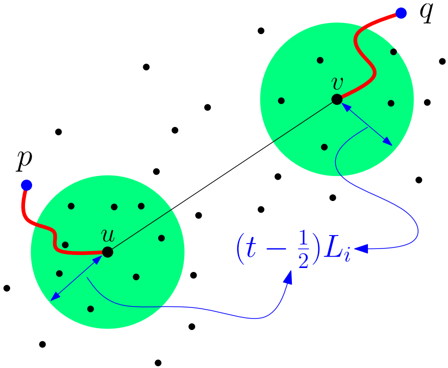
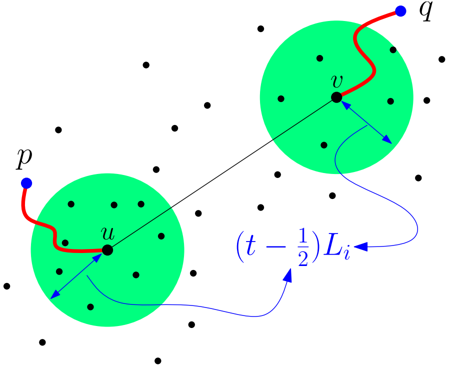


$$\mathbf{d}_G(p,q) \quad = \quad \mathbf{d}_G(p,u) + \mathbf{d}(u,v) + \mathbf{d}_G(v,q)$$

Algorithm

Updating the weight matrix after adding an edge



$$\begin{aligned} \mathbf{d}_G(p,q) &= \mathbf{d}_G(p,u) + \mathbf{d}(u,v) + \mathbf{d}_G(v,q) \\ &\geq \mathbf{d}(p,u) + \mathbf{d}(u,v) + \mathbf{d}(v,q) \end{aligned}$$

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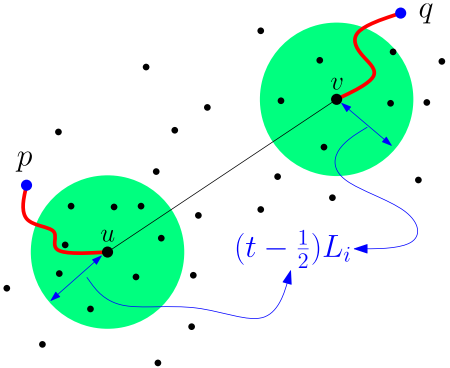

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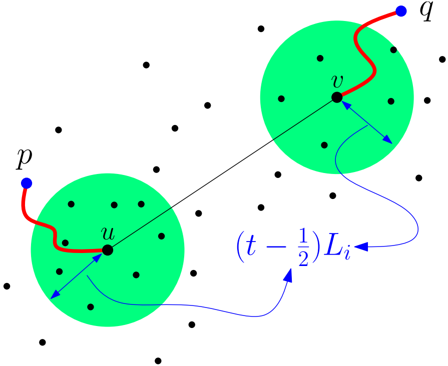

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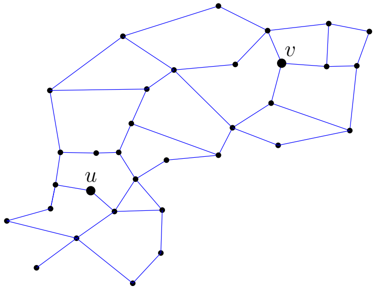
Improved Algorithm

Main ideas:

- Splitting all pairs of points into $\mathcal{O}(n)$ buckets and processing a bucket at a time.
- Keeping the weight matrix up-to-date for all pairs in processing bucket.
  - Update the weight matrix at the beginning of processing a bucket (by running bounded Dijkstra's algorithm).
  - After adding an edge, update the weight matrix (partially).
  - Instead of running Dijkstra's algorithm from scratch, we fix the previous run.

Improved Algorithm

Fixing Dijkstra



Observations:

- The Dijkstra's algorithm with source p are the same on G and G' until the key of the element on the top of the priority queue is less than $\mathbf{d}_G(p,u) + \mathbf{d}(u,v)$.
- For a fix point p, and a fixed bucket E_i, the number of times bounded SSSP (p) is computed is at most $\mathcal{O}(1/(t-1))$.

Improved Algorithm

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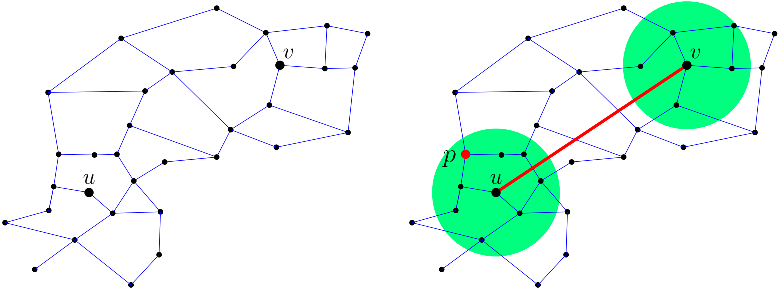

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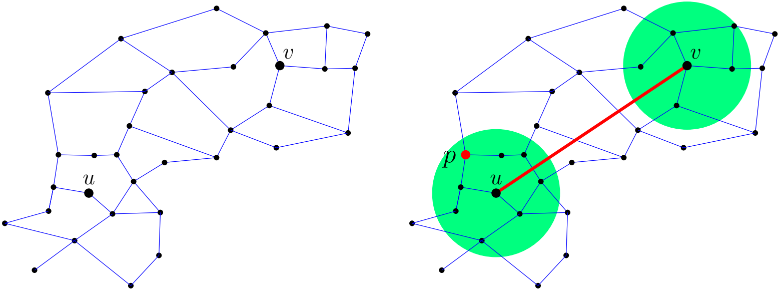

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Comments

- This results in an algorithm that uses only $\mathcal{O}(n)$ shortest path computations all together.
- This works for other metrics - but then the number of edges in the greedy graph may not be linear and the number of times shortest paths need to be computed will be proportional to that.
- The algorithm is fairly simple - uses Dijkstra's SSSP + knowing whether a point is in a disc or not.

OPEN PROBLEMS

- Is sorting all pairs of points really necessary to compute the greedy spanner?
- Can we compute greedy spanner for points in the plane under Euclidean metric in sub-quadratic time?

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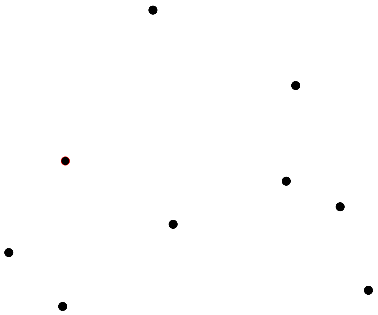
Θ-Graph Construction

- For each point p, divide the plane around p into cones, with angle $\Theta \leq \pi/4$.
- For each cone, connect p to its nearest neighbor in the cone.



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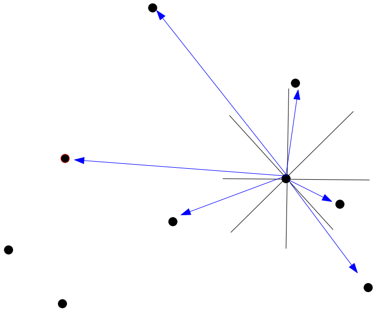

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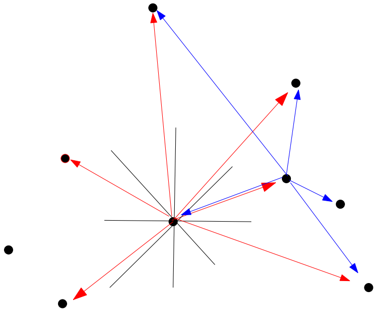
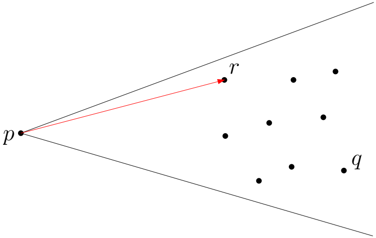




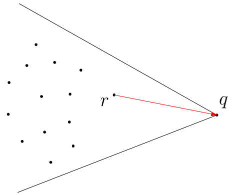
- For each p and for each θ-cone with apex p:
  - edge (p, r), where r is an **approximate** nearest neighbor of p in the cone
  - this edge takes us closer to q by at least $(\cos\theta - \sin\theta)|pr|$
  - stretch factor $t \leq \frac{1}{\cos\theta - \sin\theta} \rightarrow 1$ if $\theta \rightarrow 0$

Properties of the Θ-Graph

- **Good:**
  - time to construct: $O(n \log n)$
  - outdegree: $O(1)$
  - number of edges: $O(n)$
- **Bad:**
  - indegree: $n - 1$
  - weight: $\Omega(n \cdot wt(\text{MST}))$
  - spanner diameter: $n - 1$

q-Sink Spanner: Only Require Short Paths to a Fixed q

- θ-cones with apex q, each one containing at most $n/2$ points
- for each cone:
  - edge (r, q), where r is the nearest neighbor of q in the cone
  - r-sink spanner on the points in the cone

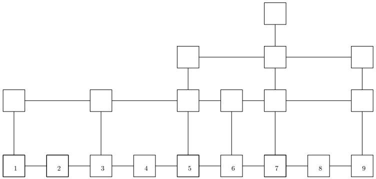


- outdegree $= 1$; indegree $= O(1)$; $O(n \log n)$ time

Variant of the Θ-Graph: Bounded Degree

- $G = \Theta$-graph (or any spanner whose outdegree is $O(1)$)
- For each vertex q:
  - let $\text{IN}(q) = \{p : (p, q) \in G\}$
  - replace incoming edges of q by a q-sink spanner of $\text{IN}(q) \cup \{q\}$
- This gives a spanner whose in- and out-degree is $O(1)$

1d-small diameter skip list spanner



Properties

- Height is $\mathcal{O}(\log n)$.
- Total size is $\mathcal{O}(n)$.
- Path has length $\mathcal{O}(\log n)$.

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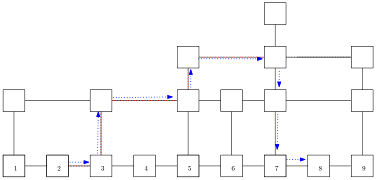

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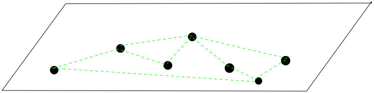
Variant of the Θ-Graph: Reduce the Spanner Diameter

- Generalize **skip lists**
- Define
  - $S_0 = S$
  - S_i contains each point of S_{i-1} with probability $1/2$
- Construct a Θ-graph for each subset S_i



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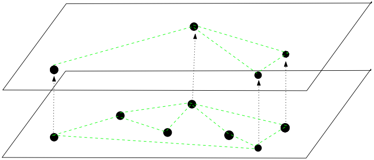

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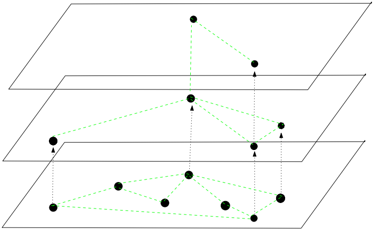

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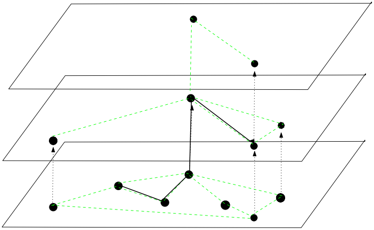

Skip-list Spanner

How to find the path between p and q?

- Follow the nose algorithm from p and q in S_0 till either the spanner path is found or the vertices p_1 and q_1 at the next level are found.
- Repeat the above step for p_1 and q_1 at the next level.
- At some level i, the path from p_i and q_i will meet at a vertex.
- Trace back these two paths all the way back to the first level to obtain the spanner path!

This gives:

- stretch factor $t \leq \frac{1}{\cos\theta - \sin\theta}$
- $O(n)$ edges (w.h.p.)
- spanner diameter $O(\log n)$ with high probability! WHY?

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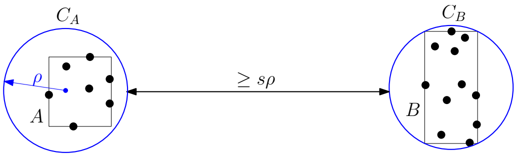
Well-Separate Pair Decomposition



Well-Separated Pair

Let $s > 0$ be a real number, and let A and B two finite sets of points, then we say that A and B are *well-separated with respect to s* if there are two disjoint balls C_A and C_B, such that

- C_A and C_B have the same radius
- C_A contains the bounding box of A
- C_B contains the bounding box of B
- the distance between C_A and C_B is greater than s times ρ

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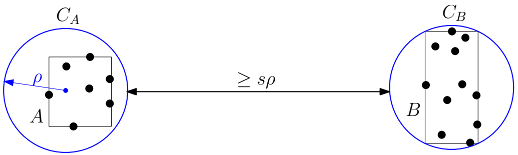

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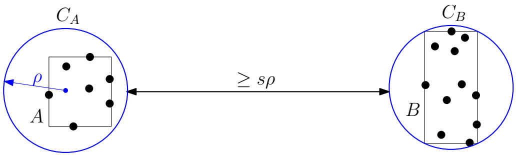

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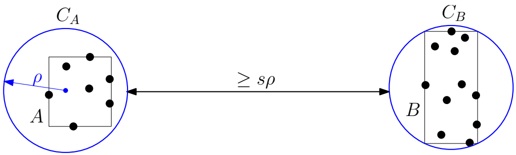

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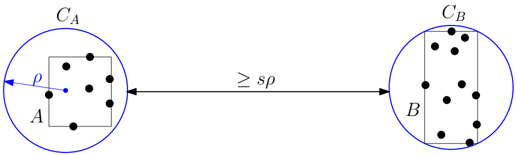

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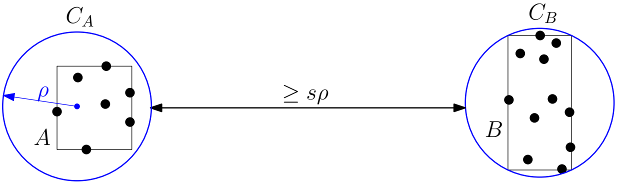
WSPD Contd.



Well-Separated Pair Decomposition with respect to s

Is a sequence $\{A_1, B_1\}, \{A_2, B_2\}, \ldots \{A_m, B_m\}$ of pairs such that

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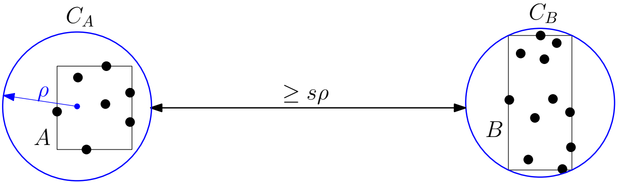

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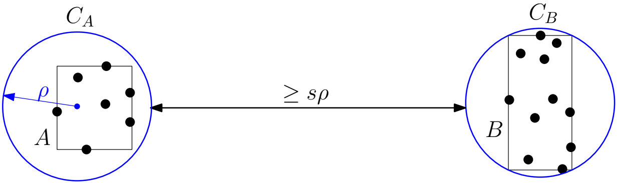

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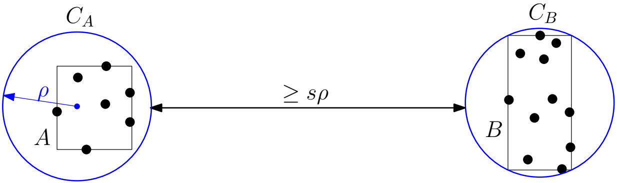

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Remarks

- Proposed by Callahan and Kosaraju in 1992.
- Essentially it says that “the number of distinct distances in the plane among a set of n-points is linear and NOT quadratic”.
- Many problems based on distances between pairs of points can be solved efficiently (e.g., n-body simulation, closest pair, all nearest-neighbors, spanners).
- Works very nicely in higher dimensions (Computation Time of WSPDs $\rightarrow \mathcal{O}(dn \log n)$. Computation Time of Voronoi Diagrams/Range trees etc. $\rightarrow \mathcal{O}(n \log^{\mathcal{O}(d)} n)$. $d =$ dimension).

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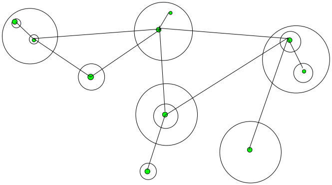
WSPD-Spanner

- Given a WSPD $\{A_1, B_1\}, \{A_2, B_2\}, \dots, \{A_m, B_m\}$
- For each i, take one edge (a_i, b_i), where $a_i \in A_i$ and $b_i \in B_i$



- This gives a t-spanner for
$$t = \frac{s+4}{s-4} \rightarrow 1 \text{ if } s \rightarrow \infty$$
- WSPD with $m = O(n)$ can be computed in $O(n \log n)$ time

WSPD-Spanner

- degree can be large, but
  - the edges can be directed such that the outdegree is $O(1)$
  - combined with sink spanners: degree $O(1)$
  - time to construct: $O(n \log n)$
- using the gap property (and more): weight is $O(\text{MST} \cdot \log n)$
- choose the edges more carefully: spanner diameter $O(\log n)$
- WSPD-spanner can be represented by $O(1)$ trees such that for all p, q, one of the trees contains a t-spanner path between p and q.
  - by adding shortcuts to these trees: spanner diameter $O(\alpha(n))$ and $O(n)$ edges

Computation of WSPD

Main Steps in Constructing WSPD

- Compute a SPLIT TREE.
- Extract well-separated pairs from the Split Tree.

Split Tree

Definition

It is a recursively defined binary tree, where each node stores a bounding rectangle of points in its subtree.

- Root stores the bounding rectangle of the whole set S.
- If $|S| = 1$, then the split tree is a single node storing that point.
- Otherwise, split the bounding rectangle of S by splitting the longest side into two halves. This splits the point sets into two - S_1 and S_2. Split tree consists of a node corresponding to S and two children corresponding to recursively defined split trees of S_1 and S_2.

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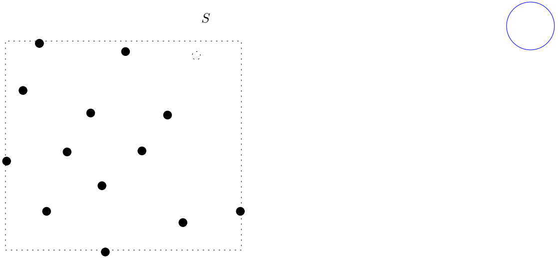

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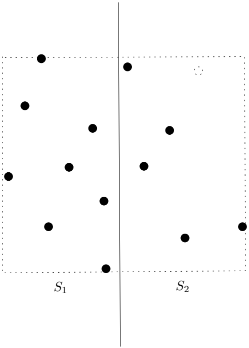

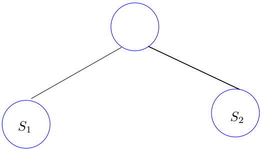

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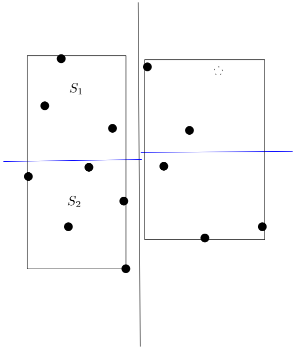

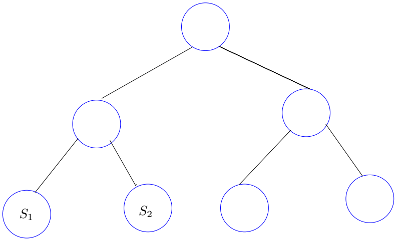

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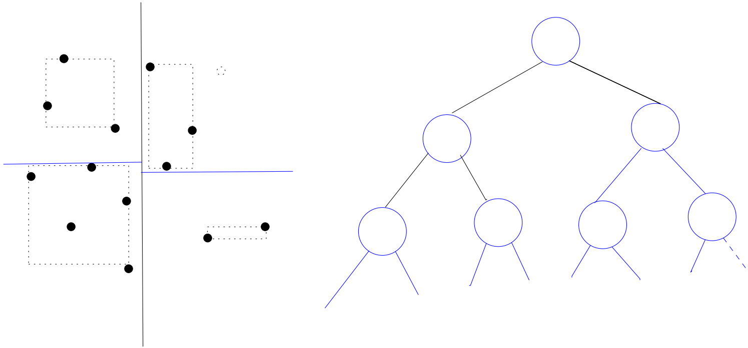

Computing the WSPD

For each internal node u of the Split Tree execute FINDPAIRS(v, w);
v is left child of u and w is the right child.

FINDPAIRS(v, w)

if S_v *and* S_w *are well-separated* **then**
  return the node pair $\{v, w\}$
end
else
  (assume that longer side of bounding rectangle of v is smaller than that of w);
  $w_l :=$ left child of w;
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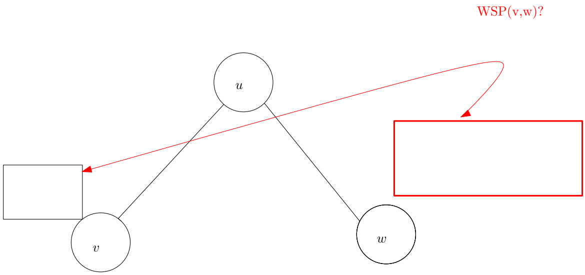

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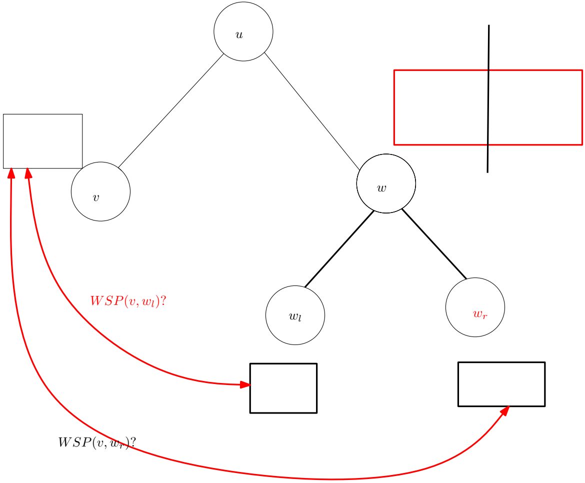

Why does it work?

Lemma

Let $\{v_i, w_i\}$, $1 \leq i \leq m$, be the sequence of node pairs returned by the algorithm FINDPAIRS. The sequence $\{S_{v_1}, S_{w_1}\}, \cdots, \{S_{v_m}, S_{w_m}\}$ is a WSPD for the set S.

Proof

Observe that each of the pairs $\{S_{v_i}, S_{w_i}\}$ is well separated and non-empty.
Consider any two points p and q in S.
Consider $LCA(p, q)$ in the fair split tree
Let $u = LCA(p, q)$ and its left and right children be v and w.
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If S_v and S_w are well separated, then $p \in S_v$ and $q \in S_w$, or we make calls to the appropriate children, and by induction on the number of recursive calls we can show that p and q are in a unique pair which is well separated.

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Why is $m = O(n)$?

Why the number of pairs returned by the algorithm is linear and not quadratic?

Where are the $\binom{n}{2}$ distances hiding?

The key in understanding this is the packing argument.

Key Observation

If (S_v, S_w) is a pair in the WSPD, then the pairs $(S_{\pi(v)}, S_w)$ or $(S_v, S_{\pi(w)})$ are not well-separated.

An Implication

Let (S_v, S_{w_i}), $1 \leq i \leq k$, be all the pairs in the WSPD that have S_v as one of the component. Then all the sets S_{w_i}'s are pairwise disjoint.

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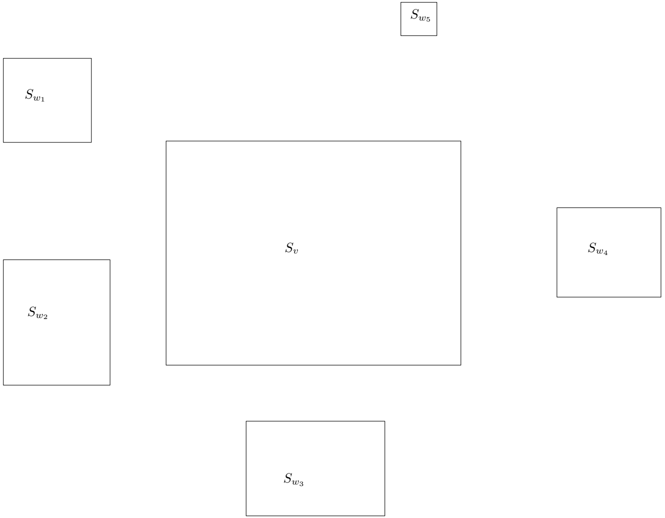

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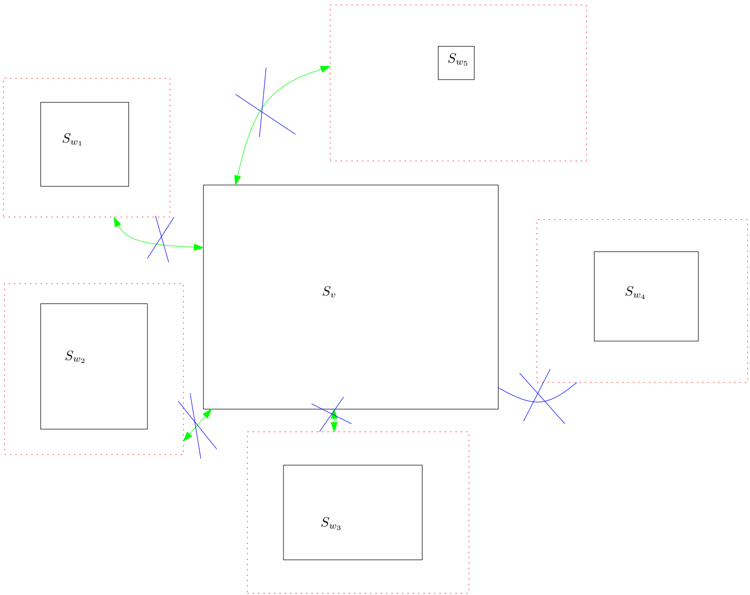

What do we have now?

Main Theorem

Given a set of n points in the plane, we can compute a well-separated pair decomposition, using fair split tree, of size $\mathcal{O}(n)$, in $\mathcal{O}(n \log n)$ time.

Corollaries

In $\mathcal{O}(n \log n)$ time we can compute

- t-spanner of a point set.
- closest pair in a point set.
- Nearest neighbor of each point in the point set.
- Approximate MST.
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Extend the concept of Well-Separated Pair Decomposition to other Metrics.

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Raman and Chebrolu in 2005 proposed a new protocol for communication

- It is not possible for one node to both transmit and receive at the same time
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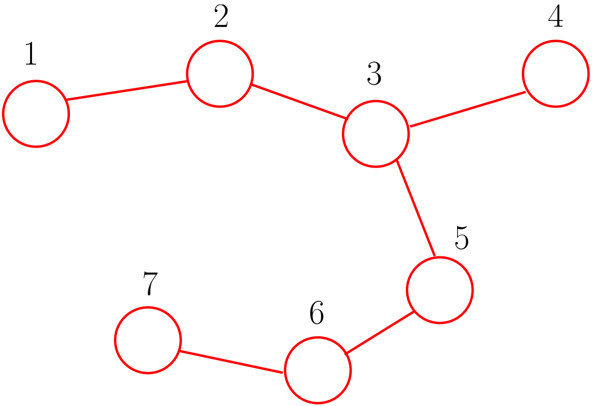

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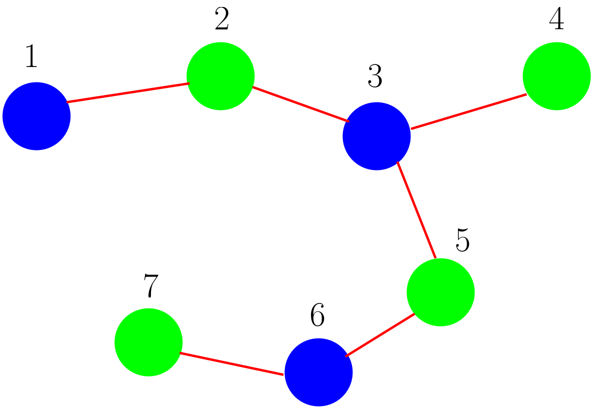

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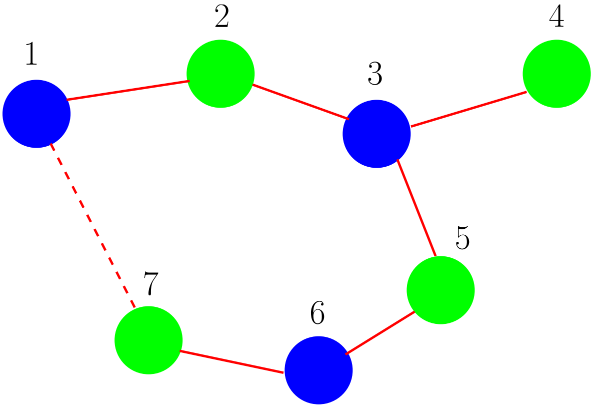
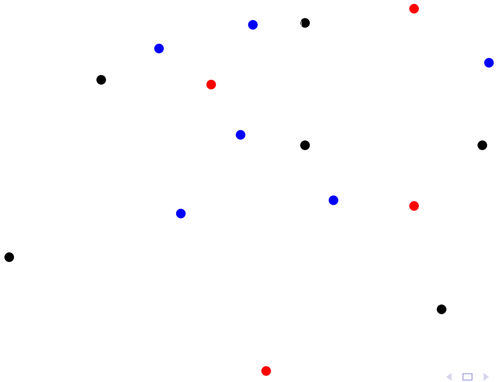


Spanners of complete k-partite geometric graphs

Problem I definition

Given a point set P and a coloring of P, compute a t-spanner for this coloring.



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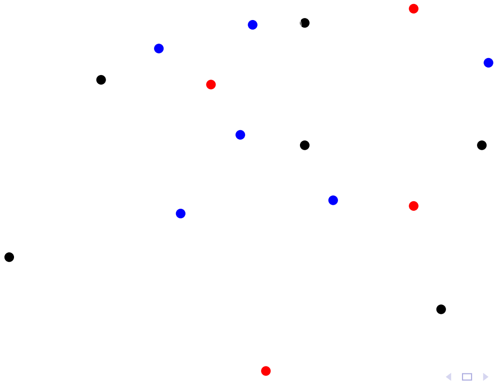

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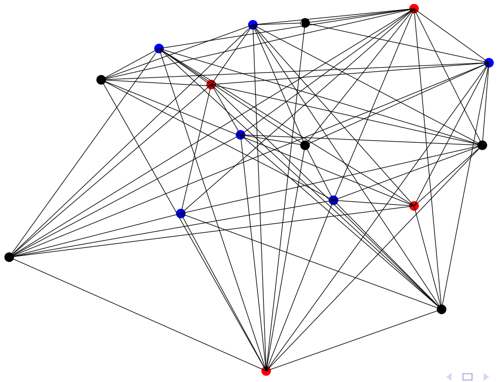

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Results

- Algorithm that computes $(5+\epsilon)$-spanner of G, with $O(n)$ edges in $O(n \log n)$ time.
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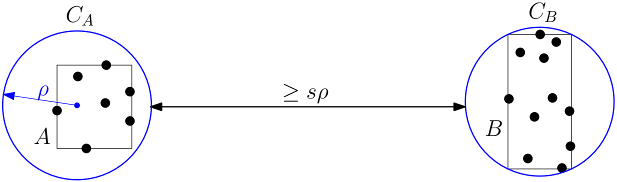
Well-Separate Pair Decomposition (WSPD) [Callahan and Kosaraju 1992]



Well-Separated Pair

Let $s > 0$ be a real number, and let A and B two finite sets of points in $\mathbb{R}$, then we say that A and B are *well-separated with respect to s* if there are two disjoint balls C_A and C_B, such that

- C_A and C_B have the same radius
- C_A contains the bounding box of A
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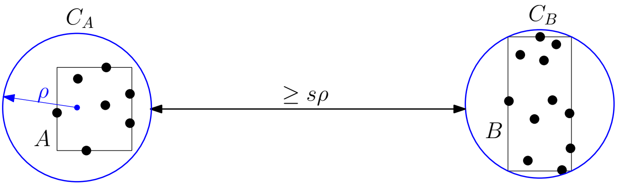

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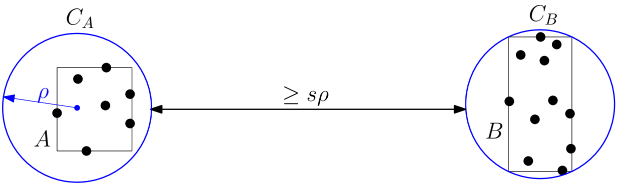

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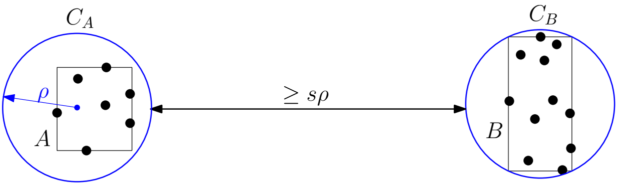

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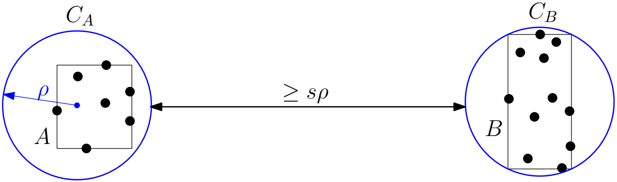

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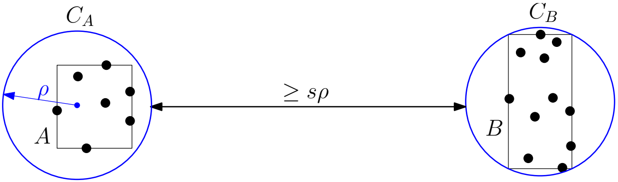

Well-Separated Pair Decomposition with respect to s

Is a sequence $\{A_1, B_1\}, \{A_2, B_2\}, \ldots \{A_m, B_m\}$ of pairs such that

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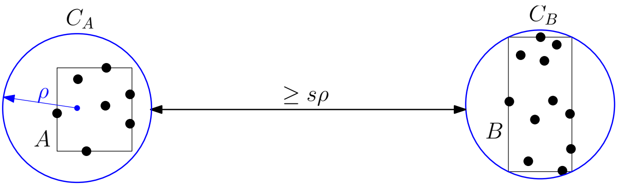

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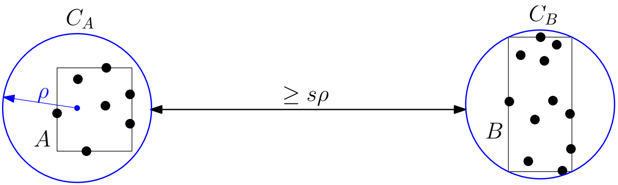

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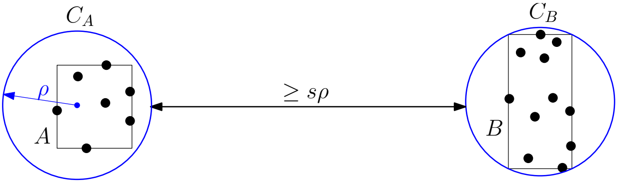

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First attempt

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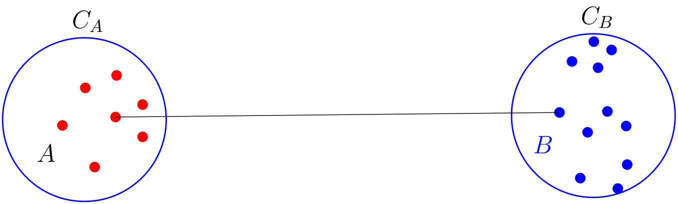
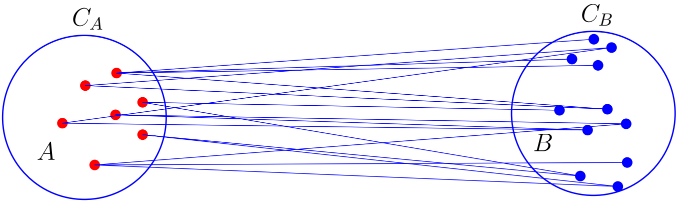


In WSPD we pick one edge

First attempt



Now we need lots of edges

Second attempt

1. compute a WSPD
2. for each pair $\{A, B\}$ do as follows
   - if both A and B are all red or all blue, then ignore

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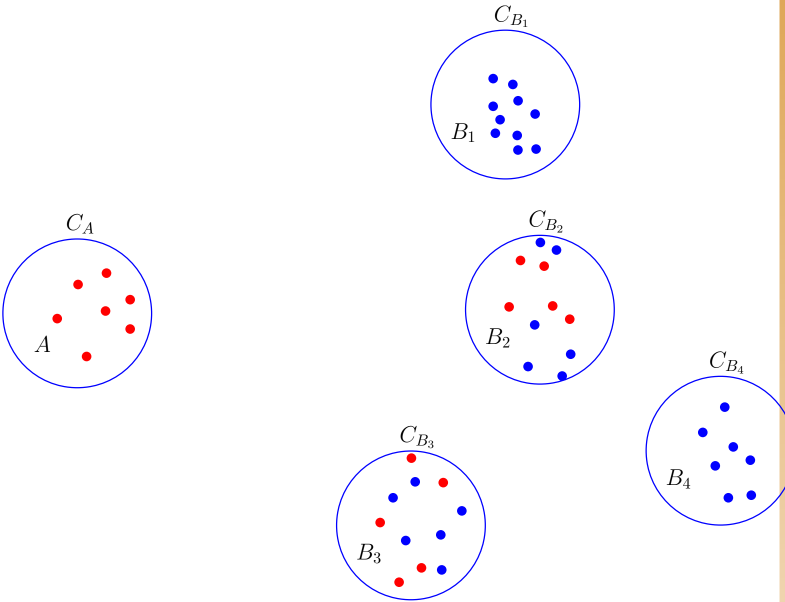

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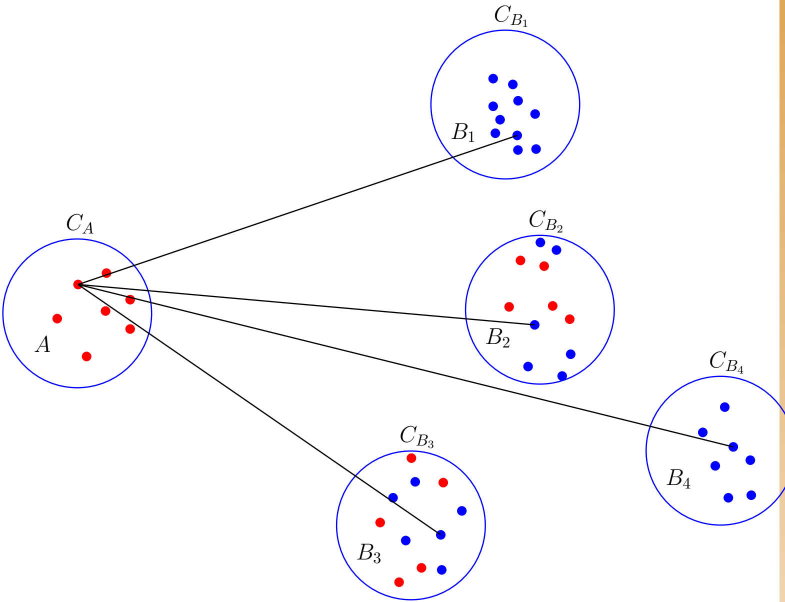

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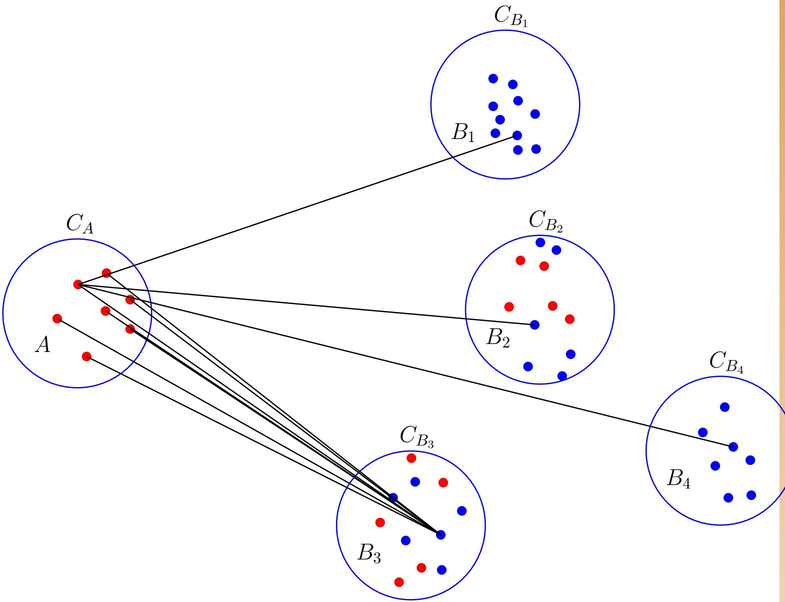
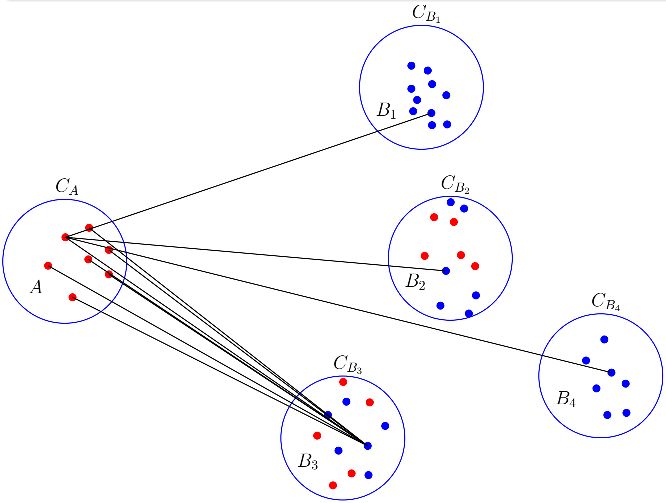
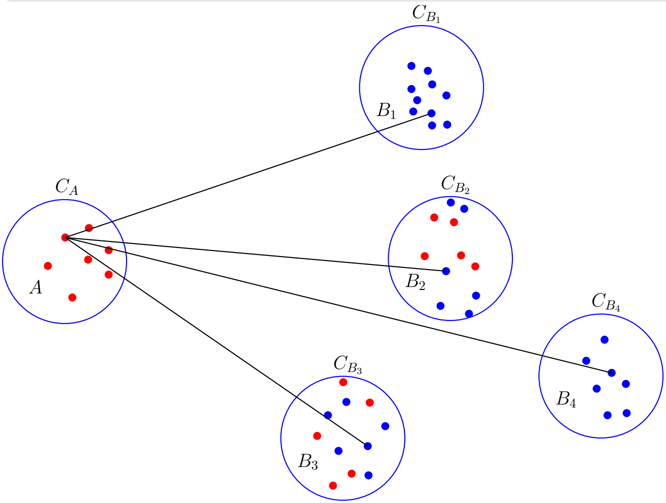


Second attempt

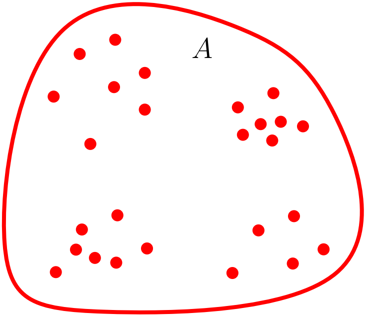
This will lead to quadratic number of edges



fixing the problem



The algorithm



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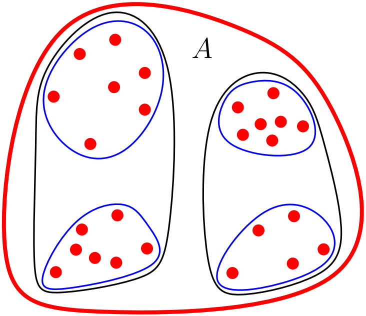

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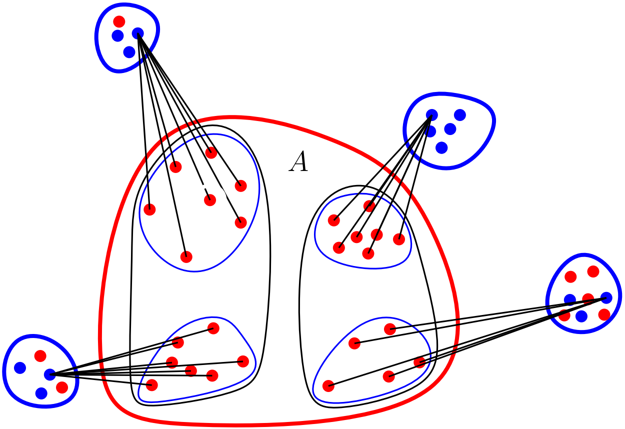

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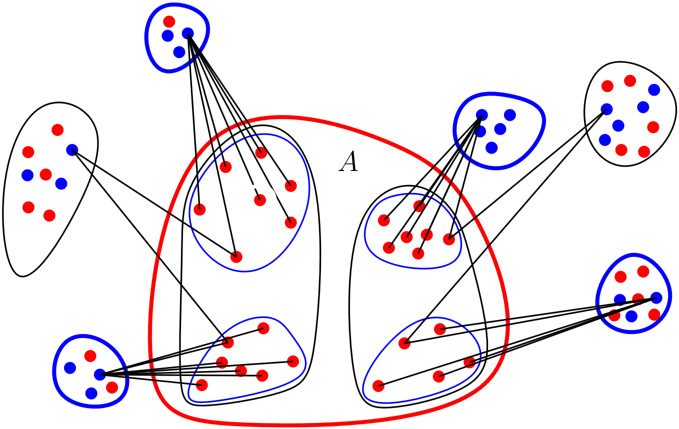

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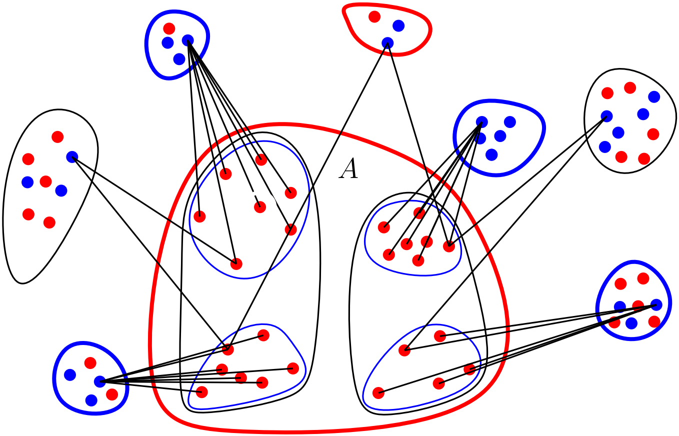

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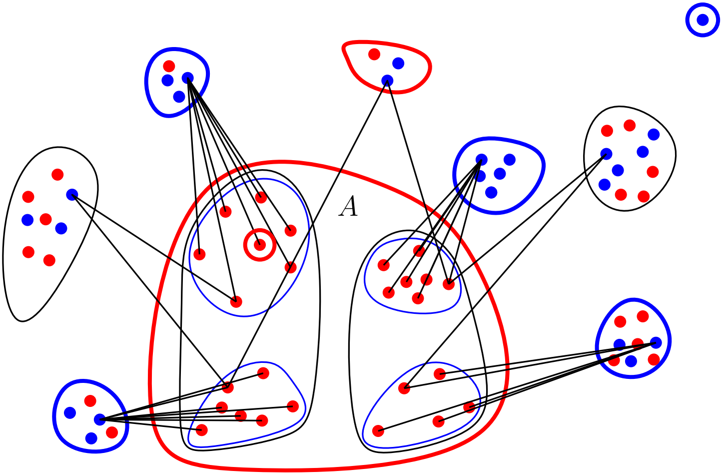

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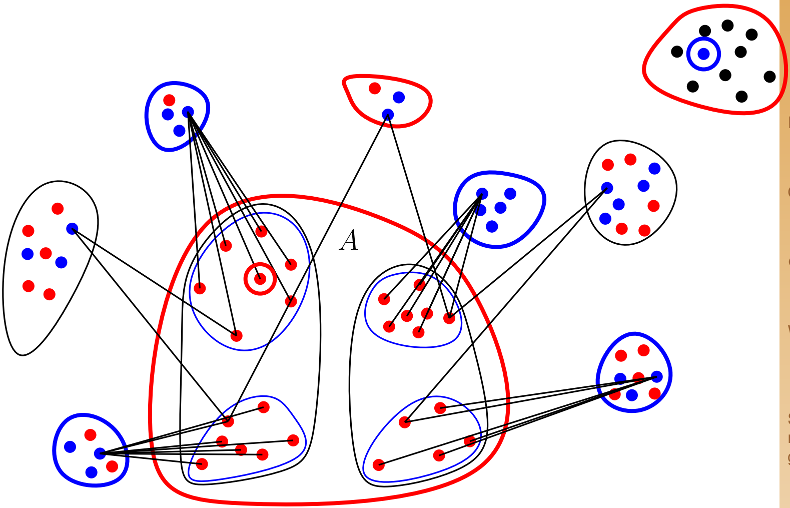

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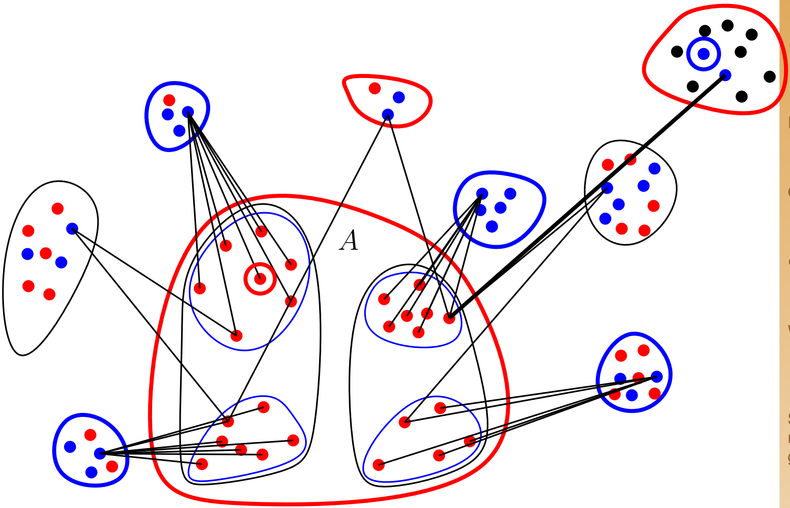

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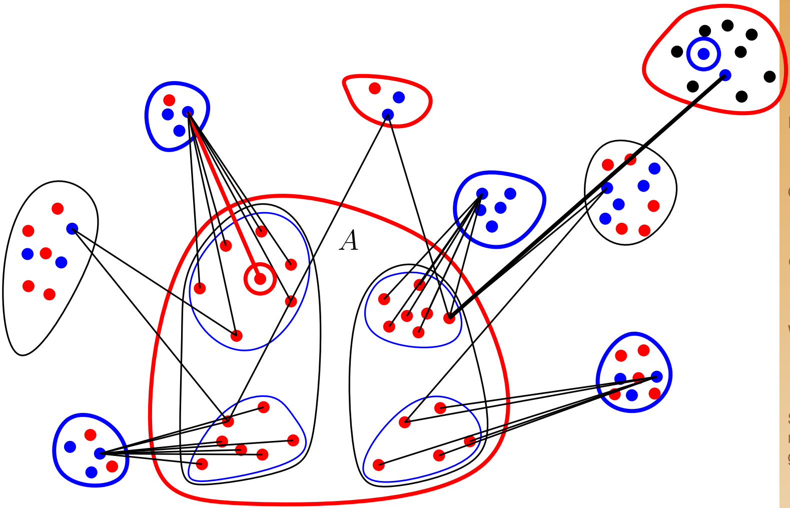

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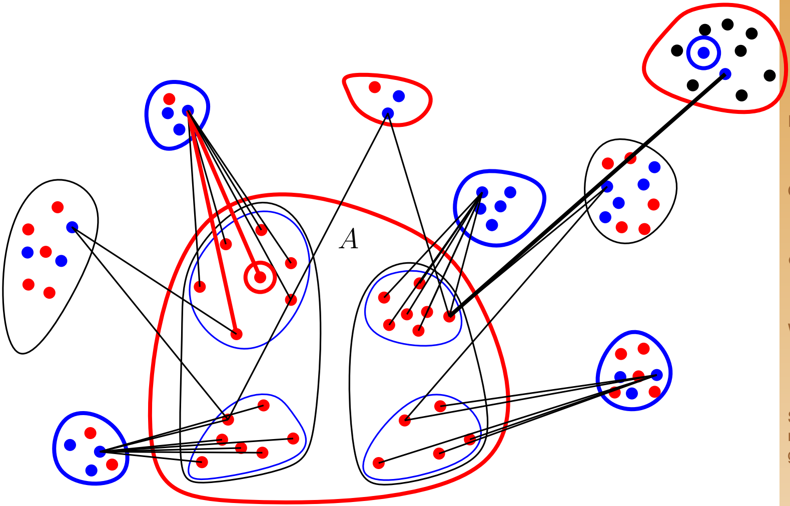

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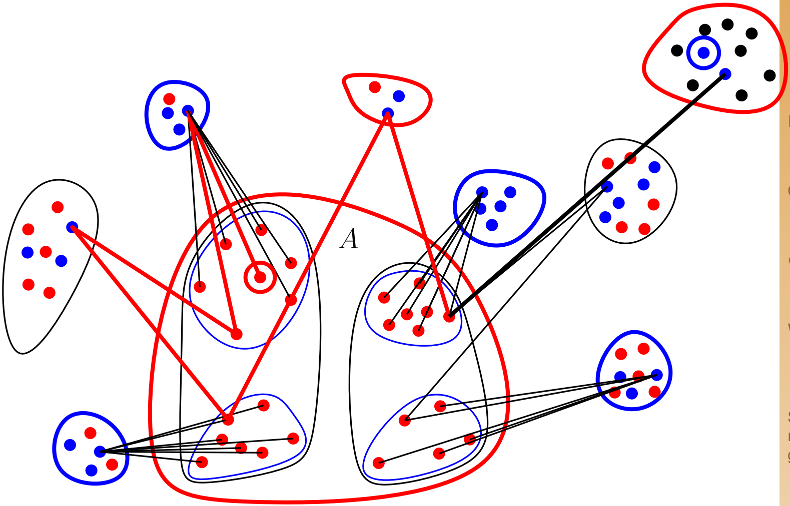

Spanners of complete k-partite geometric graphs

Summary

Given a complete k-partite geometric graph G, compute a spanner that has “small” stretch factor and “few” edges.

Results

Algorithm that computes $(5+\epsilon)$-spanner of G, with $O(n)$ edges in $O(n \log n)$ time.

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Spanners with small chromatic number

Problem II (the points are not colored)

Given a point set P and an integer $k > 1$, compute a t-spanner for P whose chromatic number is at most k.

Upper and lower bounds

Given an integer $k \geq 2$, compute the smallest value $t(k)$ such that for each set P of points in the plane,

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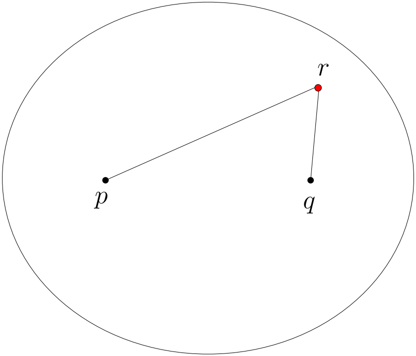
The t-ellipse property

A coloring satisfies the t-*ellipse property*

if $\forall p, q \in P$ with $c(p) = c(q)$, there $\exists r \in P$ such that $c(r) \neq c(p)$ and $|pr| + |rq| \leq t|pq|$.



The t-ellipse property

The equivalence

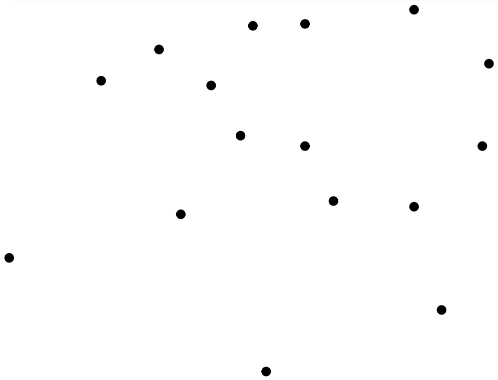
Determining the smallest value of t such that a k-chromatic t-spanner exists for any point set P

$$\equiv$$

Minimizing the value of t such that any point set can be colored using k colors in a way that satisfies the t-ellipse property

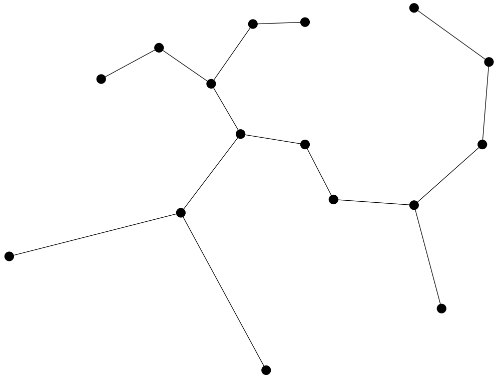
Two colors ($t(k) = 3$)

Input: P, a set of points in the plane



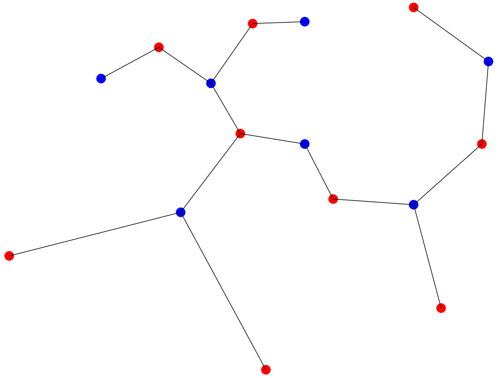
Two colors ($t(k) = 3$)

Compute an Euclidean minimum spanning tree T of P

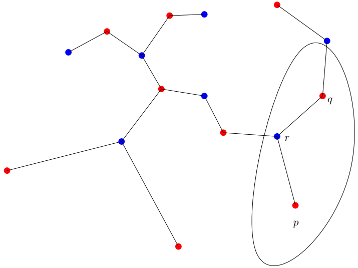


Two colors ($t(k) = 3$)

$c \leftarrow$ a 2-coloring of T



2-coloring with the 3-ellipse property



Let r be nearest neighbor of p, i.e. $|pr| \leq |pq|$.

$$Color(r) \neq Color(p)$$

$$|pr| + |rq| \leq |pr| + |rp| + |pq| = 2|pr| + |pq| \leq 2|pq| + |pq| = 3|pq|.$$

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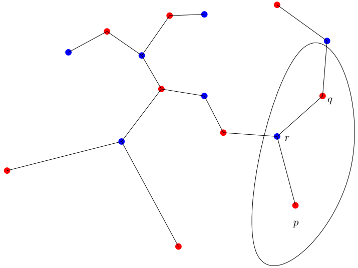

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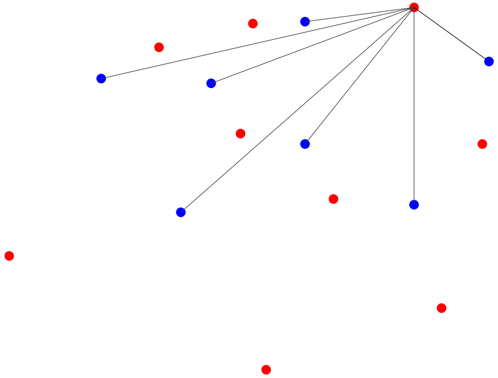
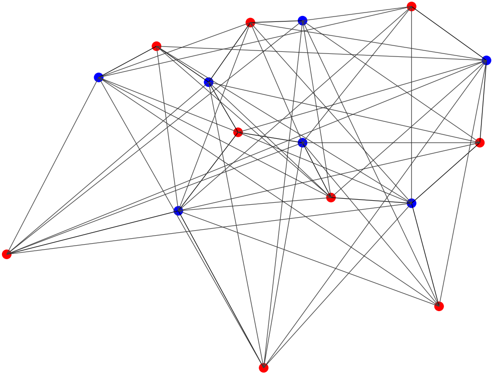
Draw an edge from each red point to all blue points



Two colors ($t(k) = 3$)

The complete bipartite graph



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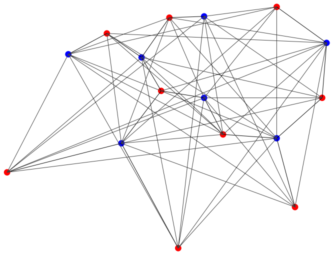
The complete bipartite graph has the 3-Ellipse property



Two colors ($t(k) = 3$)

We obtain a coloring that has the 3-Ellipse property

Theorem by J. Gudmundsson, C. Levcopoulos, G. Narasimhan, and M. Smid, 2002

For any t-spanner for P contains a subgraph with $O(n)$ edges which is a $((1+\epsilon)t)$-spanner for P.

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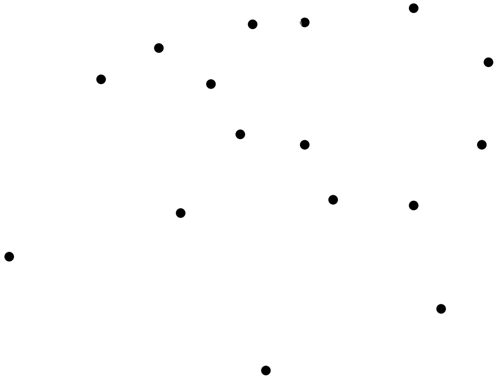
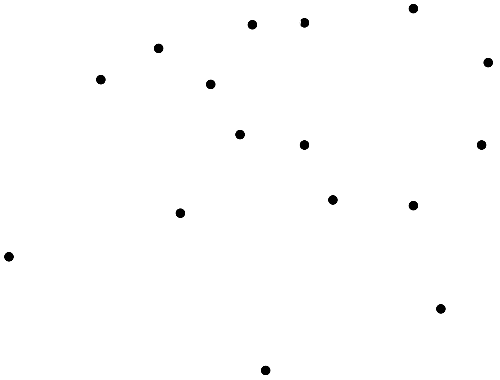


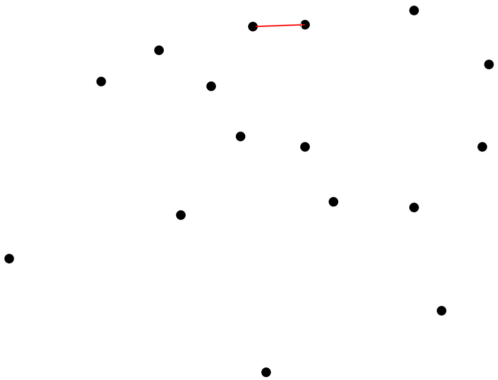
Three colors $(t(k) = 2)$

Sorting the edges of the complete Euclidean graph on P



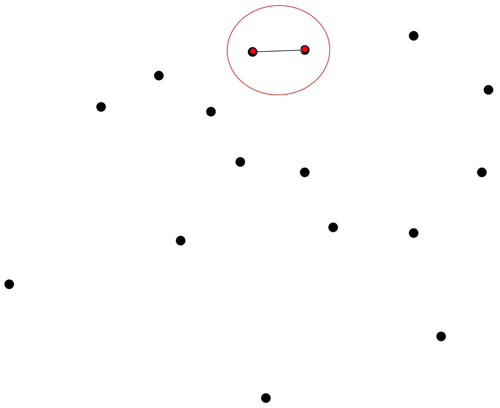
Three colors ($t(k) = 2$)

Pick the shortest edge



Three colors ($t(k) = 2$)

Add the edge if the 2-ellipse does not contain other edge



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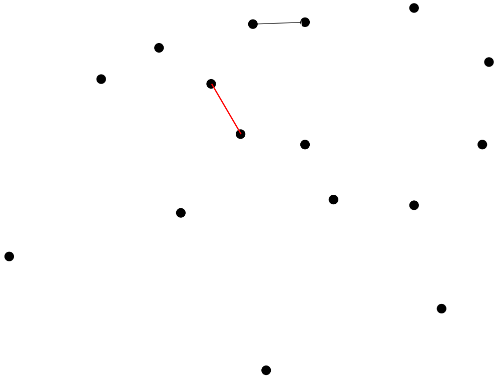
Pick the next shortest edge



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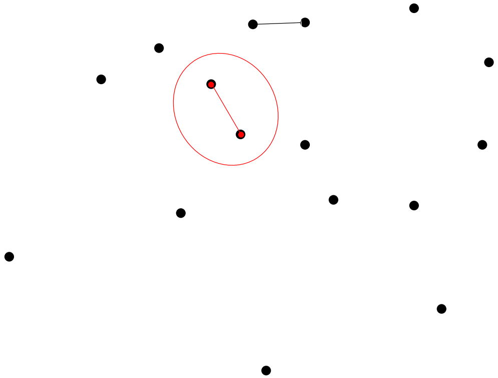

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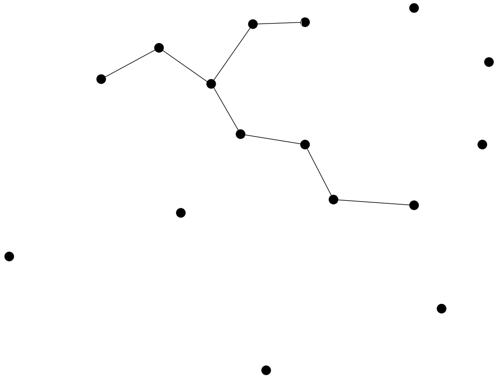

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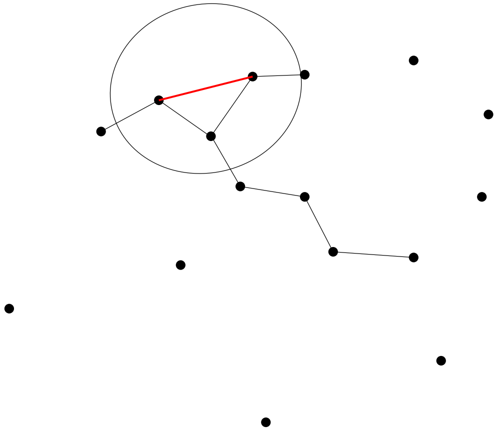

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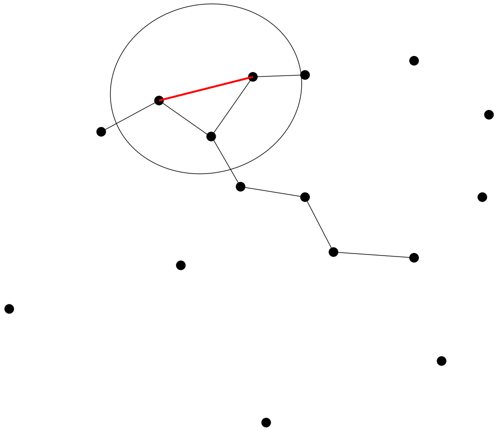
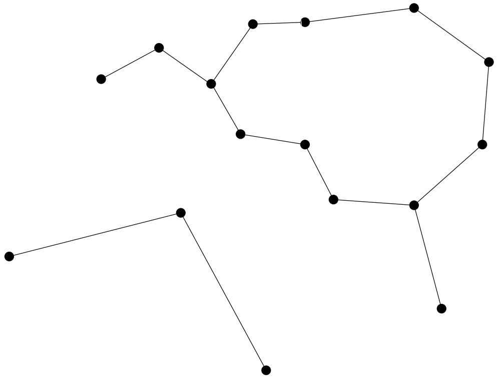


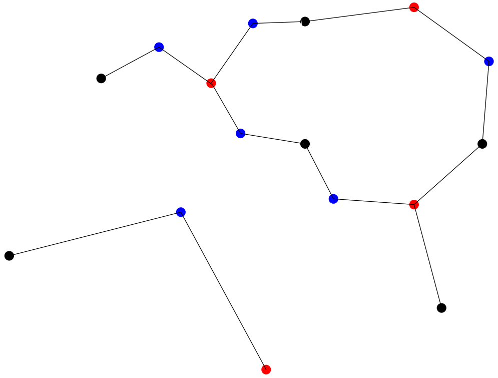
Three colors ($t(k) = 2$)

Planar graph without triangles



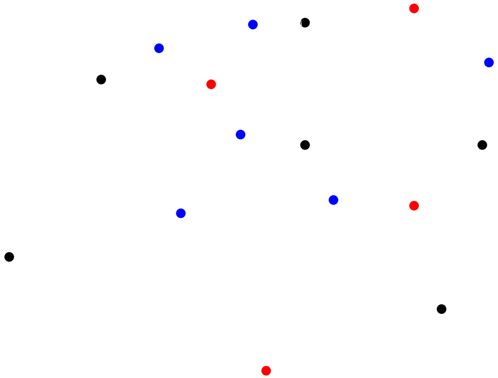
Three colors ($t(k) = 2$)

Planar graph without triangles, thus it is 3-colorable [Grozsch 1959]



Three colors ($t(k) = 2$)

3-Coloring that has the 2-ellipse property



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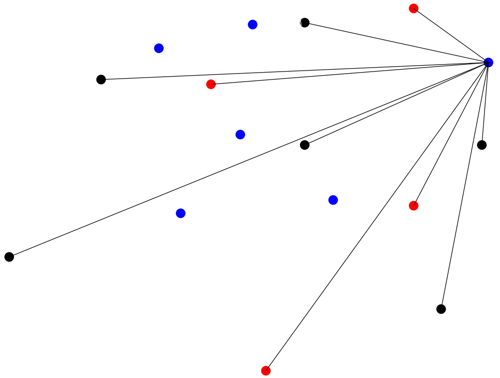
Complete 3-partite graph



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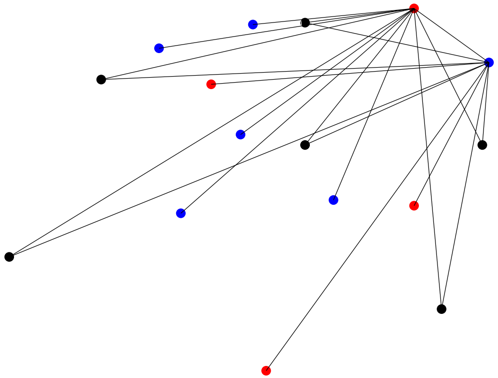

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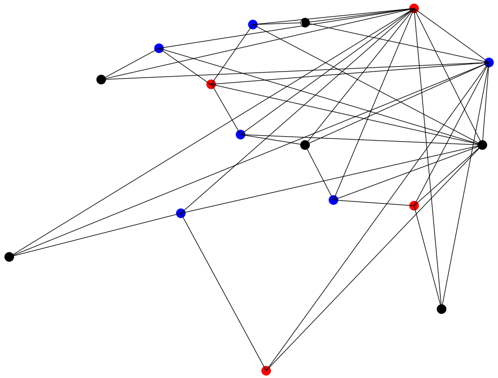

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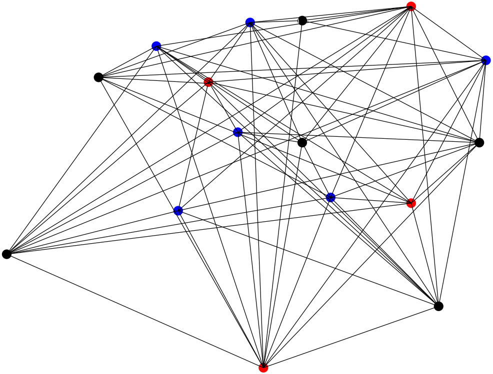
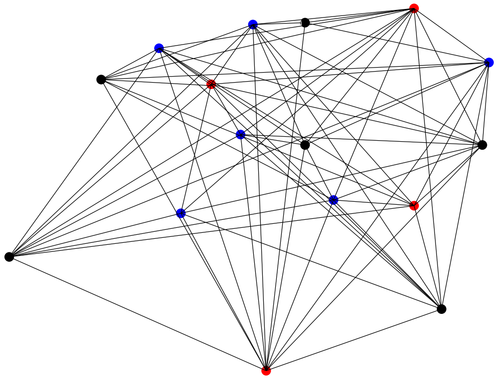
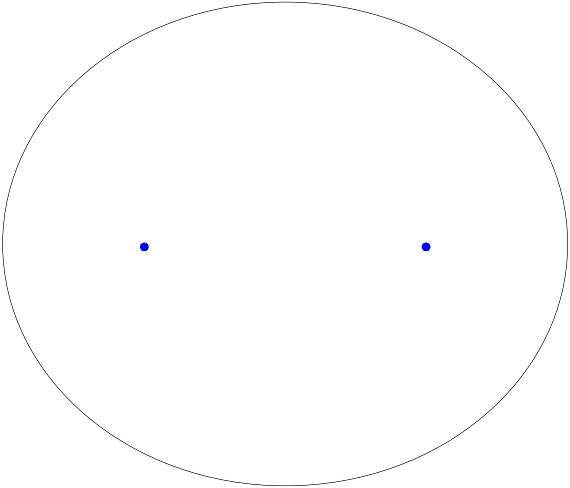


Three colors ($t(k) = 2$)

Complete 3-partite graph, with the 2-ellipse property



Three colors ($t(k) = 2$)



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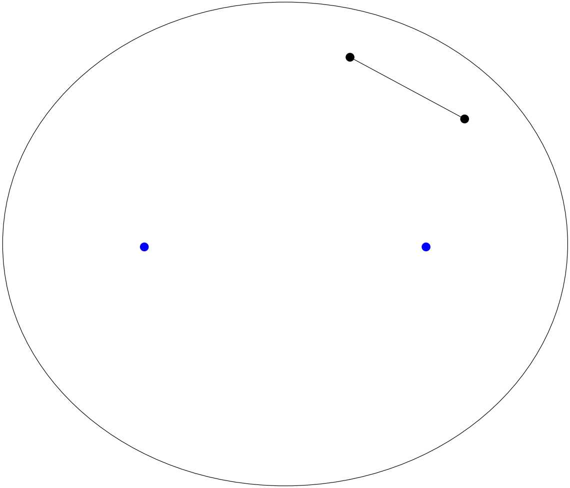

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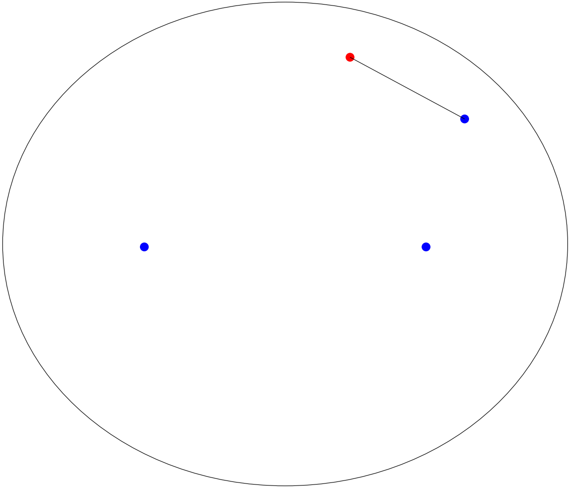

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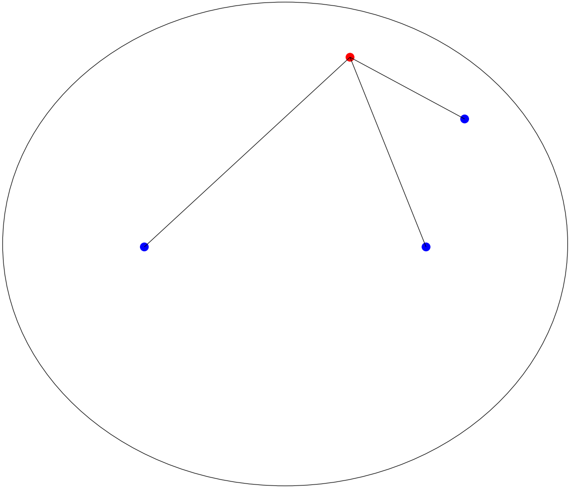

Four colors

Using the Delaunay triangulation we obtain a graph that is:

- planar
- has $(\sqrt{2})$-ellipse property

References

- Book by Narsimhan and Smid on "Geometric Spanners" published in 2008.
- Websites of: Anil Maheshwari/Michiel Smid/Paz Carmi/Prosenjit Bose
  www.cg.carleton.ca
  Carleton University, Ottawa, Canada.

Thank You.