The Philosophy of Mathematics

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(*Continued from the previous issue*)

The Mathematical Method

Having understood the nature of mathematical concepts, we now need to briefly examine the mathematical method. What is the method by which we arrive at the truth or falsity of mathematical statements?

In a mathematical system, we have axioms, which are facts taken to be obviously true ('If *a* is less than *b*, then *a* is not equal to *b*' is one such axiom—the axiom of linear order), and some non-facts (which we shall call *non-axioms*) by the help of which we prove or disprove theorems. Proving theorems means deriving them from known axioms. If we are able to deduce a theorem starting from these basic axioms, then we say that the theorem is true.

However, proving the falsity of a theorem is different. If we are able to derive a non-axiom from a proposition, then that proposition is false—a *non-theorem*. So, here we go the other way round—we start from the theorem itself, not from non-axioms. Hence proving the truth and falsity of theorems are not mirror processes.

The underlying assumption of this method is that we cannot derive a non-fact from facts. Such a system is called *consistent*. If a system is inconsistent, it is 'trivially complete'; that is, every statement, true or false, is derivable in an inconsistent system. An inconsistent system, therefore, is of little practical use.

Propositional Logic

In the study of mathematical method we also need to study propositional logic. Propositions play an important part in mathematical proofs. What is a proposition? A *proposition* is a statement which is either true or false. Note that there are certain statements which are neither true nor false. For example, interrogatory and exclamatory statements are neither true nor false. Also there is this classic example of a paradoxical self-referential statement, which is neither true nor false:

P: The statement P is false

We have referred to the term *theorem*. Now is the time to define it. What is a theorem? A *theorem* is nothing but a proposition for which there is a formal proof. What then is meant by proof? A *proof* is simply a sequence of deductive steps governed by well-defined logical rules that follow from a set of axioms. An *axiom*, of course, is a proposition that is given to be unconditionally true. The following deduction illustrates the rule of specialization, which is one of the many rules of logic:

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal

Thus, a mathematical system is a set of axioms and non-axioms with predefined rules of deduction, which are also referred to as rules of inference. The *rules of deduction* or *rules of inference* are nothing but rules that add, remove, modify, and substitute operators and symbols.

Let us try an exercise to understand how the rules of inference work. Suppose we have been given the following rules of addition, removal, and substitution of symbols I and U (the other symbol M remains there as in the starting axiom). The starting axiom is MI, and *x* and *y* are variables:

(i) *x*I → *x*IU (Derive MUUIIIU from MUUIII)
(ii) M*x* → M*xx* (Derive MUUIIIUUIII from MUUIII)
(iii) *x*III*y* → *x*U*y* (Derive MUUU from MUUIII)
(iv) *x*UU*y* → *xy* (Derive MIII from MUUIII)

Now try constructing the theorem MU starting only with the axiom MI using the above rules of inference. Is it possible to derive MU?

The crux of the matter discussed above is that in mathematics, as well as in logic, the operators, the constants, and the functions can all be viewed, as in this example, as symbols which are added, removed, and substituted by predefined rules of inference, without ascribing any interpretation to them. Gödel exploited this fact beautifully in proving his famous theorem on incompleteness.

Is Mathematics a Uniquely Human Activity?

Since doing mathematics involves intricate reasoning and abstract thinking, it is often thought to be a very creative process requiring a lot of intuition. Kant was of the opinion that since mathematics requires human intuition it cannot possibly be done by non-humans. But several later philosophers have shown that it really does not require any human intuition to understand a mathematical proof. Finding a proof for an open research problem, though, might be an altogether different matter—computers have failed till date to automatically generate proofs for even very simple non-trivial mathematical problems. This is not to suggest that proving mathematical theorems is a uniquely human activity incapable of computer simulation—it is simply a matter of selective processing power. Computers cannot distinguish between boring mathematical truths and interesting mathematical results and keep happily churning out one mathematically uninteresting result after another, *ad infinitum*.

Mathematical thinking, in fact, is apparently not unique to humans. Rudimentary mathematical understanding is also seen in other animal species. And, of course, computers are 'doing' mathematics all the time. If one is to argue that finding and discovering mathematical truths rather than understanding proofs constitutes the test of mathematical intelligence—and computers fail this test—then it may be pointed out that this will also place the majority of humans at par with machine intelligence, because the vast majority of humans do not participate in the exciting activity of mathematical discovery.

Important Branches of Mathematics

Among the important branches of mathematics, number theory, set theory, geometry, and logic are historically very old. The oldest civilizations—the Indian, Greek, Chinese, Egyptian, and Babylonian—had all developed these branches, in one form or other, for general use. This is substantiated by the fact that without a fair understanding of geometry the remarkable architectural and civil-engineering feats for which these civilisations are famous would not have been possible. Even such elementary constructions as a rectangular wall or a field, or the more intricate hemispherical dome, require at least a rudimentary knowledge of geometrical constructions. Incidentally, ancient Greeks gave much importance to geometry, whereas Indians gave up geometry for abstract mathematics during the Buddhist period.

As far as number and set theories are concerned, no one really knows when humans developed these. Numbers surely came with the need for counting. Most civilizations seem to have been formally using numbers right from their inception. It was needed for commerce, and in earlier tribal societies to quantify one's possessions.

Set theory is more fundamental than number theory, for it deals with classification rather than counting. Formal logic was a later development. But its rudiments were probably coeval with the development of language—with the need to coherently and intelligently communicate one's opinions, arguments, and deductions to others. In fact, logic and language are so interlinked that many consider logic to be merely a linguistic construct. Historically, both Nyaya and Aristotelian philosophy had formalized logic for their respective civilizations, the Indian and the Greek.

Number Theory

Let us begin with numbers. We may ask: What is the nature of numbers? Are numbers real? In the

Nyaya and Vaisheshika philosophies, for instance, numbers are real entities, belonging to one of the seven categories of real entities. However, there are conceptual difficulties if we grant numbers an objective reality. Consider the following: We have two books. So, we have books and we have also the number two. Let us add another pair of books to our collection. Does it destroy the number two and create the number four? Or does the number two transform into the number four? Suppose we add two notebooks, to distinguish them from the original pair of books. Then we have got a pair of twos as well as a four. None of the original numbers is destroyed or transformed and yet a new number is created. The ancient Buddhists were therefore not wrong in pointing out that numbers are in fact mental concepts. They do not have any existence outside the mental world.

Furthermore, mathematicians say that numbers can also be thought of as properties of sets, being their sizes (though the Buddhists would not feel comfortable with this either). Numbers as properties of sets were called *cardinal numbers* by George Cantor in contrast to *ordinal numbers* which represented positions in a series (first, second, and so on). Again, these are not to be taken as real properties, for there is an equally long-standing debate on substances and their properties. Essentially, therefore, numbers are abstract properties of equally abstract sets. Or, with greater ingenuity, the abstract concept of set itself can be thought of as representing numbers—not just the properties of sets but the sets themselves. Thus, we may have:

$\{\} = 0$

$\{\varphi\} = 1$

$\{\varphi, \{\varphi\}\} = 2$

$\{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}\} = 3$, and so forth.

Does anyone find this remarkable example illuminating or fascinating! All the same, this is what we meant by our statement that mathematical entities are not real but are merely conceptual entities.

Historically, the notion of numbers was formalized in the following succession. The notion of *natural numbers* (1, 2, 3, …) was developed first. 'God created the natural numbers; everything else is man's handiwork', the German mathematician Leopold Kronecker had famously observed. The incorporation of zero as a number was the great contribution of the Indian subcontinent. The natural numbers are complete as far as the operations of addition and multiplication are concerned—if we add or multiply two natural numbers we get another natural number. However, the class of natural numbers is not complete with respect to subtraction (you don't get a natural number if you subtract 3 from 2). So if the result is to belong to the set of numbers, we need to extend the list of natural numbers to include negative numbers. The result is the set of *integers*.

Again, we see that the class of integers is not complete with respect to division. So the set of numbers is further extended to include ratios—*rational numbers*. The word *rational* here is derived from 'ratio' and not 'reason'. Next we get surds or irrational numbers, when we extend the set of numbers to include limits, sums of series, square roots, trigonometric functions, logarithms, exponential functions, and so on. This gives us *real numbers*. Actually, this gives us only a subset of real numbers because these constitute only what are called computable numbers (which can be computed to any desired degree of precision by a finite, terminating algorithm). Not all real numbers can be so constructed. To be mathematically precise, we need to see each real number as a partition which divides the set of numbers into two groups A and B. If the partition is such that there is a largest element of A or a smallest element of B then the (partitioning) number is rational. But if there is neither a largest number in A nor a smallest number in B then the divisive number is irrational. This is the concept of 'cuts' developed by the celebrated mathematician Richard Dedekind.

The other day I was arguing with a friend that every real number can be seen as a decimal expansion which can be computed one digit after another using a suitable algorithm. I was, however, wrong. Alan Turing has proved this mind-boggling truth that not all real numbers are computable.

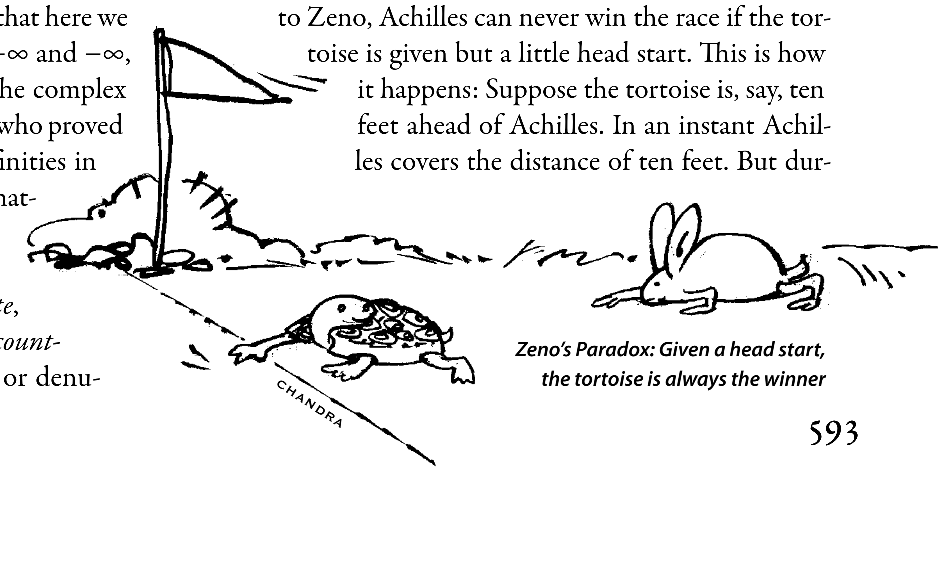
People found out very quickly that negative numbers could not have real square roots. In order to make the set of numbers complete even with the operation of determining square roots, the domain of real numbers was again extended to that of *complex numbers*, which are nothing but the sum of a *real number* and an *imaginary number* (i.e. a number expressed as a multiple of $\sqrt{-1}$). The historical choice of the names *imaginary* and *complex* was, however, unfortunate. For this makes one think that complex numbers are not numbers at all. One could on the other hand look at complex numbers as a dyad such that the subset of this dyad with the second term as zero is actually the set of real numbers. Moreover, all algebraic operations that can be carried out using real numbers can also be applied to the complex number dyads when these are suitably redefined. This interpretation is much more appropriate than the one commonly taught in schools. It is also worth noting that the class of complex numbers is 'complete' in the sense that if we apply any normal operator or any common function to complex numbers we always get a complex number.

With the introduction of complex numbers, one would think that the number system was at peace. But that was not to be, for serious trouble was brewing with the inclusion of the concept of infinity. There is a common misconception that there is one and only one mathematical infinity. And the people who seem to be more prone to this misconception are people from a Vedantic background! I wish to point out that here we are not merely thinking of $+\infty$ and $-\infty$, or even 'radial infinites' in the complex plane. It was George Cantor who proved that there are numerous infinities in relation to numbers. As a matter of fact, while the set of integers and of rational numbers are *countably infinite*, the set of real numbers is *uncountably infinite*. (Countability or denumerability refers to being able to be counted by one-to-one correspondence with the infinite set of all positive integers.) More remarkably, Cantor was able to prove that even uncountably infinite sets have different cardinalities: that if $\aleph_0$ is an infinite set then there exists a set ($\aleph_1$) which can be proved to be larger than this set, and this process can be extended to obtain infinites with still greater *cardinality*. Cantor's treatment of infinities, however, was abstract rather than constructive. And this cost him an appointment at Berlin University—though his work was mathematically sound—as Kronecker, a firm believer in constructions, opposed him. Mathematicians, after all, are also human!

Zeno's Paradox

Besides the problem of infinity, mathematicians working with numbers had also to tackle the problems of limits and series. To appreciate the problem with series, we consider one of Zeno's paradoxes—a set of problems devised by Zeno of Elea to support Parmenides's doctrine that 'all is one'. This doctrine asserts that, contrary to the evidence of our senses, the belief in plurality and change is mistaken, and, in particular, motion is nothing but an illusion. This is much like the Buddhist doctrine of *kṣaṇikavāda*.

'Achilles and the Tortoise' is the most famous of these paradoxes. Fleet-footed Achilles, of Battle-of-Troy fame (in Homer's *Iliad*), and a tortoise are participating in a race. Achilles is reputed to be the fastest runner on earth; and the tortoise is one of the slowest of living beings. However, according to Zeno, Achilles can never win the race if the tortoise is given but a little head start. This is how it happens: Suppose the tortoise is, say, ten feet ahead of Achilles. In an instant Achilles covers the distance of ten feet. But dur-



Zeno's Paradox: Given a head start, the tortoise is always the winner

ing that instant the tortoise has already advanced a short distance. Again in another bound Achilles covers that small distance, but to his dismay, during that time the tortoise has advanced still more, and so on. Thus, Achilles can never possibly catch up with the tortoise.

But this clearly is nonsense. In reality, things never happen like that. This is actually a graphic description of the problem of the sum of an infinite series of decreasing terms which yields a finite value. Of course, not every such series will yield a finite value. The harmonic series (1 + ½ + ⅓ + ¼ + ...) is one such.

Set Theory

Now that we are on paradoxes, let us start our discussion of set theory with Russell's paradox. In set theory, we have finite sets as well as infinite sets. For infinite sets it is possible that a set contains itself. {φ, {φ}, {φ,{φ}}, {φ,{φ},{φ,{φ}}}, ...} is one such set. Keen observers would have noted that this is the number 'infinity' in the 'illuminating' example of a previous section. Now call a set *abnormal* if it contains itself. Define a set *R* of all 'normal' sets: 'the set of all sets that do not contain themselves as members'. Now ask the question: Is *R* normal or abnormal? We see that this question cannot be answered in either the affirmative or the negative.

The 'axiomatic set theory' was developed to address such paradoxes by incorporating an 'axiom of choice' within the theory. But this goes beyond the scope of our discussion, although it may be mentioned in passing that a surprising corollary to this theory is the fact that a *universal set*—the hypothetical set containing all possible elements—does not exist.

In practice, sets are normally related to groups and collections of objects in the external world. Here too, a similar question, as with numbers, arises: are sets real? In Indian philosophical thought too, the same question appears repeatedly. The Buddhists, for instance, argue that the axe which is a combination of the handle and the blade does not exist 'in itself'. It is absurd, they say, to call an axe a family heirloom of great value if its blade is changed *just* five times and its handle *just* fourteen times.

This question of absurdity, however, does not arise in mathematics because sets as well as their constituent members are all hypothetical entities—conceptual objects which are granted no intrinsic reality.

Geometry

In contrast to sets and numbers, it is easy for us to see that geometrical objects are conceptual. But it was not so for the Greeks—they took their geometry seriously exactly for the opposite reason: they thought geometry was real.

Take, for instance, the case of a point and a line in a plane. What is a point? A point, as every schoolchild knows, is a geometrical object that does not have any length or breadth (all its dimensions are zero). And what is a line? A line is a geometrical object that has only length but no breadth. These very definitions make it obvious that true points and lines cannot exist in the real world distinct from our mental constructions.

Credit goes to Euclid for formalizing the field of geometry into a body of axioms and theorems. Though his treatment of the subject was fully conceptual, it took a really long time—two thousand years—for people to see that these concepts do not quite match the real world. All this time everyone had been mistakenly assuming that the world is Euclidean. Geometrical results seemed to fit our experiential world so very nicely that people failed to see that they could be unreal. Nevertheless, with the advent of Einstein's theories of relativity—both special and general—the realization dawned that the world is in fact non-Euclidean; it is more accurately described in terms of several Riemannian (or elliptic) geometries.

Another point to note is that, in formalizing geometry, we try to arrive at proofs which do not appeal to our intuition or visual sense but are logically correct. For though original mathematical insights are often derived through intuition, these 'insights' also run the risk of being proved wrong. Even the great Euclid—though he was well aware of this and therefore tried very hard to avoid intuitive judgements—himself committed a few mistakes in his proofs, because these proofs relied on the way he drew the illustrations. All the same, this does not take away any of the credit due to him in recognizing what is correct mathematical procedure. And certainly the momentous task of formalizing the great body of geometry already known at his time was not an easy task by any standard.

Logic

Mathematical logic is the final edifice of mathematics. And every logical system has to deal with the question of *completeness* and *consistency*. Completeness means that every true statement must be verifiable, must have a proof. Consistency is slightly different: it means that we should not be able to 'prove' false statements as true, that is, false statements must not have valid proofs in the theory in question.

At the beginning of the twentieth century, David Hilbert posed the ultimate problem of logic to mathematicians—to prove the consistency of mathematics as a system. This challenge came to be fondly called the Hilbert programme. Hilbert observed:

> When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps. Upon closer consideration the question arises: Whether, in any way, certain statements of single axioms depend upon one another, and whether the axioms may not therefore contain certain parts in common, which must be isolated if one wishes to arrive at a system of axioms that shall be altogether independent of one another.
>
> But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to axioms: To prove that they are not contradictory, that is, a definite number of logical steps based on them can never lead to contradictory results.

The questions of consistency and completeness are important because if mathematics as a system were both complete and consistent, then it could well yield an easy path to new discoveries by way of a method to automatically discover mathematical theorems, what with superfast computers with super-memory and super processing power as tools.

Kurt Gödel, however, proved that mathematics is in fact incomplete. He further showed that the consistency of mathematics cannot be proven from within the field of mathematics itself, or to be precise, from within Peano's axiomatization of the number theory. So with this dual stroke he delivered a terrible blow to the human quest for 'knowing everything'.

In brief, Gödel's theorems have the following twin consequences: First, there exist true statements which do not have any proof, and second, even if we have a proof for such a statement, we do not also know (by means of a valid proof) that its converse is not true. The wording and formulation of the second part is important as it makes a distinction between the truth of a statement and having a proof thereof.

A question may naturally arise at this juncture: Is Gödel's incompleteness theorem applicable to every logical system? Turing is credited with extending the results of Gödel's theorem to the field of computation. He has shown the non-existence of several kinds of computational procedures that could have helped us circumvent the implications of Gödel's theorem, enabling us to find the truth and falsity of statements in a circuitous way. Thus, he was able to draw our attention to the far-reaching consequences of Gödel's incompleteness theorem. In short, this theorem brings under its purview every kind of logical system—ancient or modern or postmodern—that is powerful enough to deduce facts. It only leaves out trivial theories like those based on first-order predicate calculus (logic).

So it would not be correct to say that Gödel's incompleteness theorem applies only to formal logic or axiomatic mathematics, and not to the Nyaya or Buddhist logical systems, because these systems also involve predicates and possess deductive (*anumāna*) power.

Mathematics, Mind, and Maya

Let me conclude with some personal reflections:

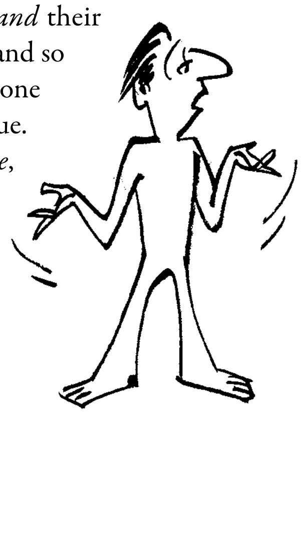
First, mathematics has to constantly fight off utilitarians who accuse it of a lack of concern with reality—at least pure mathematics does not concern itself with applications. In fact, many pure mathematicians think that applied mathematics—being more interested in the results than in the process—is a degradation, and hence no mathematics at all.

It is a mundane fact that less-advanced disciplines further their cause with the assistance of more advanced ones. The latter, however, can keep advancing only by keeping intact their pristine purity. Thus even though others may use mathematics, mathematics stands to lose if it starts catering to the demands of other disciplines: the only way for mathematics to advance is by concentrating on its lofty aims. Thus it should be left to other disciplines to find the applications for and uses of mathematics, so that pure mathematics remains pure.

Second, Gödel was able to prove that there exist true theorems for which there is no proof. Some take this as proof of the superiority of the human intellect—after all, we know indirectly about the truth of these theorems even though they cannot be proved. This is not correct. Gödel only showed that *both* the theorems *and* their converse have no proof, and so if a system is consistent, one of them is bound to be true. Thus we have, *by inference*, a true theorem which does not have a proof. But we do not know specifically which of the two (the theorem or its converse) is true. A further corollary to his theorem is that only inconsistent systems are trivially complete. And our hopes of omniscience are further dampened when we remember that the consistency of a system is impossible to prove from within the system itself.

Third, Vedanta as a system of philosophy is an empirical system. However, the only empirical facts that it sticks to with heart and soul are the reality of Brahman, the unreality of *samsara*, and the oneness of Atman, the individual soul, and Brahman, the supreme Reality. These are empirical truths according to Vedanta because Vedanta firmly holds that Atman, Brahman, and maya are mere statements of facts—a posteriori truths, truths that need to be experienced or realized. However, as the world is granted only a conceptual reality—as a construct of the cosmic mind (*hiranyagarbha*)—Vedanta remains within the purview of empirical sciences only very loosely. Strictly speaking, then, Vedanta as a system with a single composite empirical fact—*brahma satyaṁ jaganmithyā jiva brahmaiva nāparaḥ*; Brahman is real, the world unreal, and the individual soul is no different from Brahman—which is not provable by sensory perceptions, becomes a system independent of physics and mathematics alike. Nevertheless, care should be taken, when we talk (as Vedantists) either about the world that is a product of maya or when we use a deductive process to infer the unity of existence and the unreality of the world, for then there is no escape from the sciences, both empirical and formal—physics and mathematics. Within the realm of maya, Vedanta cannot go against the findings of physics and mathematics. PB

'Vedanta cannot go against the findings of physics and mathematics!'



'Only in the realm of maya!'