# *The Philosophy of Mathematics*

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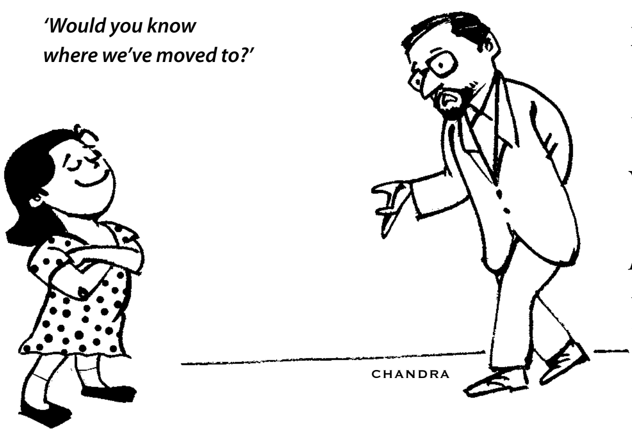
> 'Ah! then yours wasn't a really good school,' said the Mock Turtle in a tone of great relief. 'Now at OURS they had at the end of the bill, "French, music, AND WASHING—extra."'
>
> 'You couldn't have wanted it much,' said Alice; 'living at the bottom of the sea.'
>
> 'I couldn't afford to learn it,' said the Mock Turtle with a sigh. 'I only took the regular course.'
>
> 'What was that?' inquired Alice.
>
> 'Reeling and Writhing, of course, to begin with,' the Mock Turtle replied; 'and then the different branches of Arithmetic—Ambition, Distraction, Uglification, and Derision.'
>
> —*Alice's Adventures in Wonderland*

The following story is told about the reputed mathematician Norbert Weiner: When they moved from Cambridge to Newton, his wife, knowing that he would be absolutely useless on the move, packed him off to MIT while she directed the move. Since she was certain that he would forget that they had moved and where they had moved to, she wrote down the new address on a piece of paper and gave it to him. Naturally, in the course of the day, he had an insight into a problem that he had been pondering over. He reached into his pocket, found a piece of paper on which he furiously scribbled some notes, thought the matter over, decided there was a fallacy in his idea, and threw the piece of paper away. At the end of the day, he went home (to the old Cambridge address, of course). When he got there he realized that they had moved, that he had no idea where they had moved to, and that the piece of paper with the address was long gone. Fortunately inspiration struck. There was a young girl on the street and he conceived the idea of asking her where he had moved to, saying, 'Excuse me, perhaps you know me. I'm Norbert Weiner and we've just moved. Would you know where we've moved to?' To this the young girl replied, 'Yes Daddy, Mummy thought you would forget!'

*'Would you know where we've moved to?'*




The world of mathematics is beautifully reflected in *Alice's Adventures in Wonderland*—a world of ideas, where absurdity is a natural occurrence. Mathematics takes us to a world of ideas away from the ordinary, so much so that the archetypal mathematician is typified by the absent-minded professor. In fact, the world of mathematics is an imaginary world, a creation of brilliant minds who live and thrive in it. Mathematics, and the mathematicians who live in its abstract world, alike create a feeling of unworldliness in the common mind. Mathematics is itself abstract; more so is the philosophy of mathematics—the subject of the present article.

## *Why Study the Philosophy of Mathematics?*

Before we enter the subject, we must answer some questions: What is the utility of studying the philosophy of mathematics? And what specifically is the utility in the context of a journal dedicated to Vedanta?

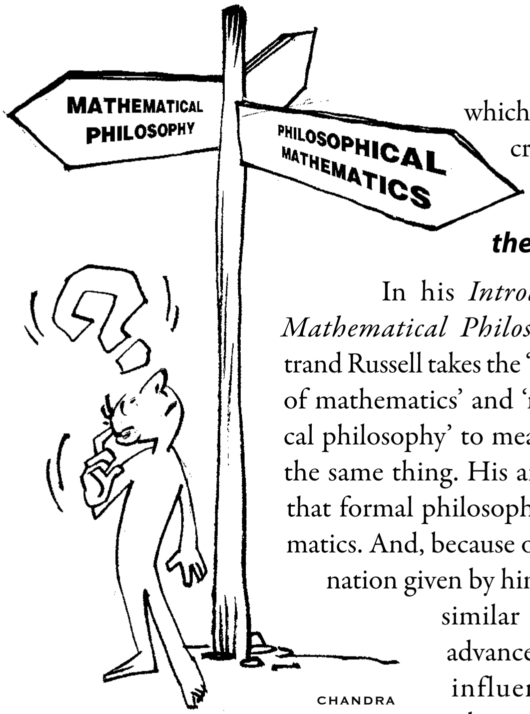
The word *philosophy* is derived from the Greek *philo-sophia*, 'love of wisdom'. Thus, in essence, philosophy as a subject tries to supplement our knowledge by finding out what is knowable and what is not; just as, in essence, logic as a subject deals with what is provable and what is not, ethics with what is right and what is wrong, aesthetics with what is beautiful and what is ugly, and religion with what is good and what is evil. Vedanta deals with what is real and what is unreal, and asserts *satyam-shivam-sundaram* as a triune entity—that which is real is also good and beautiful. So if we view Vedanta from this angle, then it is religion, ethics, aesthetics, and philosophy—all rolled into one.

The philosophy of mathematics deals with metaphysical questions related to mathematics. It discusses the fundamental assumptions of mathematics, enquires about the nature of mathematical entities and structures, and studies the philosophical implications of these assumptions and structures. Though many practising mathematicians do not think that philosophical issues are of particular relevance to their activities, yet the fact remains that these issues, like any other issue in life, do play an important role in shaping our understanding of reality as also in shaping the world of ideas. This is attested to by the fact that both the ongoing scientific revolution and the concomitant phenomenal rise of technology borrow heavily from the progress in mathematics—a dependence that can be seen throughout the evolution of civilization by the discerning mind.

The importance of mathematics can be judged by the fact that it is used in every walk of life—and this is no overstatement. It is invariably present wherever we find the touch of rational thought. It is the ubiquitous guide that shapes and reshapes our thoughts and helps us in understanding ideas and entities, both abstract and concrete. Moreover, the foundations of mathematics are rock solid. Never has a mathematical position needed retraction. Even in physics, considered a glamorous field in present-day society due to its numerous applications, one finds scientists backing out from positions they held some years earlier. But it is not so in mathematics. Once a mathematical truth is discovered, it seems to remain a truth for eternity. Why is this so?

Contrary to common belief, the real importance of mathematics does not rest in the fantastic theorems discovered; it is in the way mathematics is done—the mathematical process or methodology. It is this that is the matter of our careful scrutiny. Physics has its own methodology too, which is of equal importance. Though it may not appear obvious, both streams stress equally their respective methodologies more than the laws, theories, and hypotheses—that is, the content of physics or mathematics—that they discover or propound. That is one of the chief reasons why there is no crisis in scientific circles when one scientific theory fails and another takes its place.

Contrast this with the philosophies of old, particularly those which were not based on the firm foundation of logic. There the methodologies, the facts and theories, the lives and teachings of the proponents, and, to a lesser extent, the mythologies and cosmologies, were so intermingled, with no clear cut demarcations between them, that systems stood or fell as a whole. It was a favourite technique of opposing schools of thought to point out a single fallacy or discrepancy somewhere in a gigantic work: that was enough to invalidate the whole philosophy. Seen in this light, the strange method of proving the supremacy of one's philosophy that is often seen in Indian philosophical dialectics—through intricate and abstruse arguments as well as ludicrously naïve squabbling—is not likely to surprise us. There will be much to gain if we incorporate the logic of mathematics and the methodology of physics into our classical philosophies, and give up the esoteric dependence on classification, enumeration, categorization, and obfuscation. We need both the fine edifice of logic and the firm foundation of methodology, because most of the Indian darshanas are not mere speculative philosophies but are also empirical—they have many elements of philosophical realism. Of course, the contribution of the Indian philosophies in the realm of mind and abstract thought is enormous. Equally important are the bold proclamations of the rishis about consciousness and transcendental realities,




which are beyond criticism.

### Defining the Term

In his *Introduction to Mathematical Philosophy*, Bertrand Russell takes the 'philosophy of mathematics' and 'mathematical philosophy' to mean one and the same thing. His argument is that formal philosophy *is* mathematics. And, because of the explanation given by him as well as similar arguments advanced by other influential people, traditionally, works on mathematical philosophy also deal with the philosophy of mathematics, and vice versa. But a more commonsensical differentiation between these terms may be made thus: Mathematical philosophy is essentially philosophy done mathematically, hence falling within the purview of mathematicians, whereas philosophy of mathematics deals with the philosophical issues in mathematics, something that is to be done by philosophers. Philosophy of mathematics, as we treat the subject in this article, is indeed philosophy taking a look at mathematics, and therefore is not the same as mathematical philosophy.

Thus, we shall only try to look at answers to abstract questions related to mathematics—the form, language, and content of mathematics; the nature of mathematical concepts; and the truth and reality of mathematical discoveries and inventions. Philosophy of mathematics, hence, is truly the metaphysics of mathematics—*meta-mathematics*, the higher knowledge of mathematics. 'Normal mathematics', on the other hand, deals with the relatively mundane, the concrete, the useful, and the visible.

### The Subject Matter

Let me clarify a misconception. We are apt to think that when we talk about the philosophy of mathematics we are dealing with all that is abstruse and complicated. Nothing can be further from the truth. It is the simple facts and elementary theorems of mathematics that pose the greatest difficulty to philosophical understanding, by virtue of their fundamental nature, a nature with essential properties which we unknowingly take for granted. To illustrate the point, we list here some of the questions that philosophy of mathematics examines and the classical philosophical domains to which they belong:

- Are numbers real? (Ontology)
- Are theorems true? (Rationalism)
- Do mathematical theorems constitute knowledge? (Epistemology)
- What makes mathematics correspond to experience? (Empiricism)
- Is there any beauty in numbers, equations, or theorems? (Aesthetics)
- Which mathematical results are astounding, elegant, or beautiful? (Aesthetics)
- Is doing mathematics good or bad, right or wrong? (Ethics)
- Can non-human beings do mathematics? (Philosophy of Mind)
- Can machines do mathematics? (Artificial Intelligence)

It is customary to consider philosophical theories like mathematical realism, logical positivism, empiricism, intuitionism, and constructivism when studying the philosophy of mathematics. But we shall try to steer clear of these murky depths here.

### Nature of Mathematics

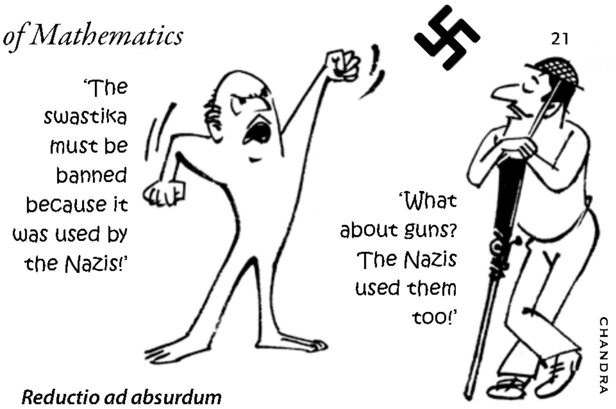
Mathematics is a formal and not empirical science. What is a formal science? A formal science endeavours to extract the form from a given piece of deductive argument and to verify the logic on the basis of the validity of form, rather than directly to interpret the content at every step. Thus, a favourite technique to prove the fallacy of an argument is to substitute hypothetical axioms in its form so that it leads to an obvious absurdity—*reductio ad absurdum*.

Another important distinguishing feature of a formal science such as mathematics is the use of the deductive method in its arguments, unlike empirical sciences such as physics which use the inductive method to arrive at generalizations.

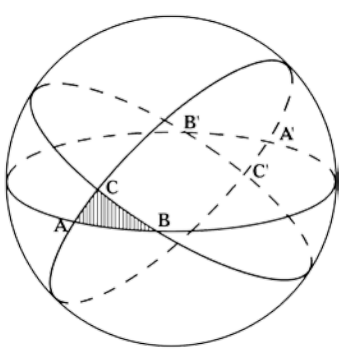
### Nature of Mathematical Entities

Are mathematical entities real? If they are not real, then whatever name we choose to call them by—*abstract* or *conceptual*—the fact remains that they exist only in our mind, a figment of our imagination—not unlike our feelings, though possibly a bit different.

It is common to acknowledge only the second possibility—that mathematical objects are definitely conceptual entities. But what does the word *conceptual* mean here? *Conceptual*, with respect to mathematical entities, means that they are hypothetical—they may or may not have any correlation with reality. In that case, these entities could be represented and interpreted in any number of ways. This fact has surprising consequences. For example, if numbers are represented by some well-structured sets—as we shall do in the section on number theory—and the operations addition, subtraction, multiplication, and division are redefined for these sets, then the sets themselves may be thought of as numbers without any loss of generality. Yet another example is that of spherical geometry. The lines of the Euclidean plane can be thought of as equatorial circles and points as poles on a spherical surface without any loss of understanding. Only the operations on lines and points will have to be redefined so that Euclidean axioms still hold true.




*Reductio ad absurdum*




*Spherical geometry: Three equatorial circles forming a triangle ABC; the sum of three angles here is not 180°*

But what are the consequences of mathematical entities being conceptual?

### On Concepts being Hypothetical

The mathematical universe consists of conceptual objects alone. There is no direct relation between mathematical entities and the phenomenal objects of the empirical world. And these mathematical objects are only indirectly correlated to existent objects and interpreted as such by the human mind. For a given system of mathematical truths, we try to interpret factual truths of the external world in such a way that they fit the mathematical model we have developed. And it may not be possible to match every mathematical model with some external reality. In other words, our mathematical models and external objective reality are connected only by our interpretation of the model. Nevertheless, it is worth noting that there is no one-to-one relationship between these two domains. There can be different mathematical explanations for the same event and, conversely, there can also be different physical interpretations of the same model. This is illustrated in the example below where we try to model real-world addition. Let us define two operators $P$ and $Q$, such that we have the following relations:

- 1 $P$ 1 $Q$ 2
- 1 $P$ 2 $Q$ 3
- 2 $P$ 3 $Q$ 5, and so on.

Given the above axioms, the operators $P$ and $Q$ could be interpreted as *plus* and *equal to* respectively; thus 1 plus 1 equals 2. Are other interpretations of $P$ and $Q$ possible? Yes. $P$ and $Q$ may also be interpreted as *equal to* and *subtracted from* respectively so that 2 *P* 3 *Q* 5 could be read as 2 equals 3 subtracted from 5. Again, *Q* could also be interpreted as *greater than or equal to* instead of *equal to*, in which case the above statement would read 2 plus 3 is greater than or equal to 5. In each of these cases we have a reasonable interpretation of the axioms, though the different interpretations of the operators *P* and *Q* are not mutually compatible. Thus we see that the same model can be interpreted in three different ways.

I always like comparing this triad of the mathematical model, the objective world, and our interpretation to that of *śabda*, *artha*, and *jñāna*—word, object, and meaning. The mathematical model of the world is equivalent to *śabda*, the world to *artha*, and the interpretation to *jñāna*. This is the way in which mathematical concepts relate to the objects of experience through an interpretation of events that is entirely a product of our thinking.

### *Mathematics and Physics*

Let us now compare the theories of mathematics and physics. What we first notice is that mathematical truths are necessary truths, that is to say, truths deducible from axioms, and true in each and every alternative system (or universe) where the axioms hold. In other words, mathematical truths are true by definition and not incidentally. Immanuel Kant, the celebrated German philosopher, called them *a priori truths*. Empirical truths, on the other hand, are *a posteriori truths*, only incidentally true. All physical facts are, surprisingly, only incidentally true. They may not be true in an alternative world or in an alternative physical system.


***Abstract thinking: Is doing mathematics an exclusively human trait?***

For example, take the speed of light. Physicists tell us that the speed of light is a constant, nearly 300,000 km/sec. Now why should the speed of light be this value? Can it not be a different value? Would the physical world appear different if the speed of light were different? When we say that 'The speed of light is nearly 300,000 km/sec' is not a necessarily true statement, then we mean that we can postulate, without fear of any technical objections, another universe where the speed of light is different, say, 310,000 km/sec. Of course, that world would be unlike ours and is not known to exist, but this line of thinking gives us a hint that there is no a priori reason for physical constants to have the immutable values that characterize them—however real they may be for us. In fact, Vedanta boldly proclaimed a long time ago that the physical universe does not have any a priori reason for its existence, and Buddhist thought has also followed this great tradition.

Here it may be of interest to draw a comparison with Nyaya, the traditional Indian system of logic. Nyaya is an empirical philosophy and is fully imbued with realism. Therefore, in its traditional five-step syllogism (*pañcāvayava anumāna*), it is mandatory to cite a real-life example (*dṛṣṭānta*) while drawing an inference from given premises. This step is much like deducing a specific instance from a general principle. And because of this thoroughly realistic approach, postulating a hypothetical universe within Nyaya discourse is virtually impossible, because that would lack real-world examples. In the mathematical domain, on the other hand, every entity is hypothetical, and entities get connected to the real world only through the interpretations applied to them. So we can postulate a hypothesis anytime and anywhere. Though Nyaya too, as a system of formal logic, has its own hypothetical concepts, its grounding in the real world restricts its conceptual flexibility. Hence, Nyaya as a logical system is able to deduce only a subset of the truths which mathematical logic is able to derive. (*To be concluded*)