Helly's Theorem and Centre Point

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Outline I

Outline II

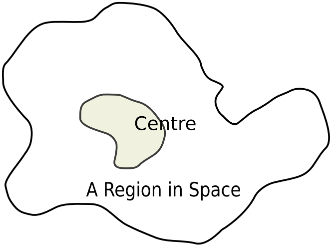
- Computation of Centre-point in Plane
- Approximate Centre-points
- Checking if a point is in Centre

5 Some Generalisations

6 Summary

Motivation for Centre Points

- We all have a notion of centre.



- Can we formalise the informal notion of centre?

Various Notions of Centre

- Like circumcentre of a triangle — minimise the maximum distance to all points.
- Like orthocentre/in-centre of a triangle — maximise the minimum distance to the exterior.

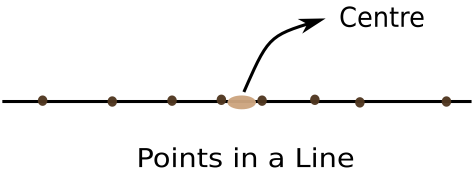
Both the above notions generalise badly. We need to generalise the notion of median of n values.

Property We are Looking for in Centre

- If possible, equal count/area/volume on *?all sides?* of centre
- Otherwise, balancing these as much as possible

Median is Centre in 1D

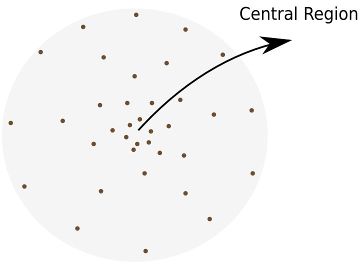
1. What is *median* of real numbers?



2. We are not interested in distances — But only in number of points in either side.
   So, we do not worry about distance metric.
3. Median partitions the set in two equal sized halves.

Centre of Uniformly Distributed points in 2D

Now suppose we have points distributed in plane.



Uniformly Distributed Points

Here centre is very nicely situated in the middle.
We ask, is it possible to find a *balanced* centre? *always*?

What do we mean by a *Balanced* Centre?

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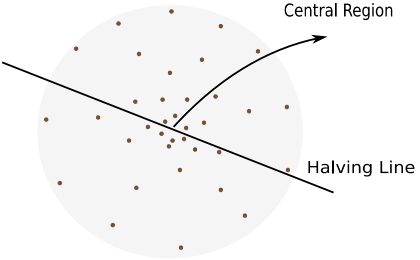


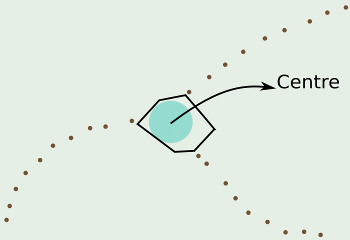
Uniformly Distributed Points

Every line through centre divides the set in half.
So, again, is it possible to find a *balanced* centre in all cases?

Worst Case Distribution in 2D

Example

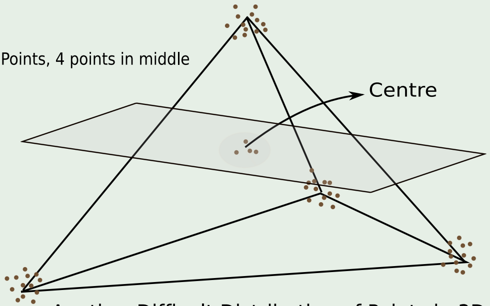
- Median divides points in equal halves.
- But in 2D, the answer is *NO*. Following is a figure where, Centre-point divides points only in one-third.



A Difficult Set of Points

Worst Case Distribution in 3D

Example



Another Difficult Distribution of Points in 3D

Centre point *dividing* points only in one-fourth.

Definition of a Centre-point

Definition

A *centre-point* of a finite set of points, P, is a point such that every closed half-space containing it contains at least $\lceil \frac{n}{d+1} \rceil$ points of P (See [7, 5, 3, 2]).

Remarks on the Definition of Centre-point

We will concentrate only in E^d, even though centre-points exists elsewhere.
What is important is the notion of half-space. Not all geometrical spaces may have this notion.

Definition of a Centre-point in Lower Dimensions

1. In 1D — Centre-point is a median point partitioning set in half.
2. In 2D — Centre-point partitions set in one-third at least.
3. In 3D — Centre-point partitions set in one-fourth at least.

Questions We can Raise

- Do centre-points always exist?
- Are they unique?

The Answers

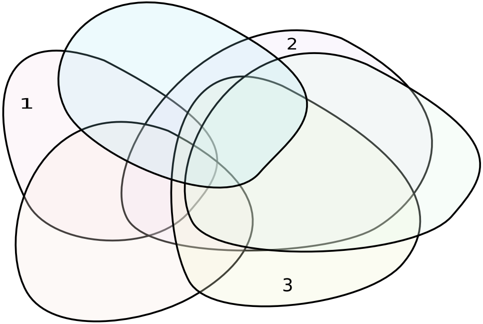
- Do centre-points always exist?
  Yes. However, we need to supply a proof.
- Are they unique?
  No. The examples above are sufficient to show that there may be many centre-points, some better and some worse.

Task Before Us

How do we prove the existence of centre-point (at least in E^d) according to the definition given previously?

Motivation for Helly's Theorem

We need to take help of Helly's theorem to prove the existence of centre point.



Six Convex Sets, Every Three of Which are Intersecting

Helle's Theorem

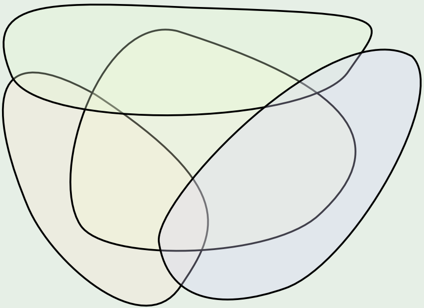
Theorem (Helly's Theorem)

Let S_1, S_2, …, S_n be $n \geq d+1$ convex sets in E^d. If every $d+1$ of the sets have common intersection than all the sets have a common intersection (See [2]).

Bound $d + 1$ is tight – I

Example

- Counter Example in $d = 2$ when we guarantee only every d sets have common intersection.

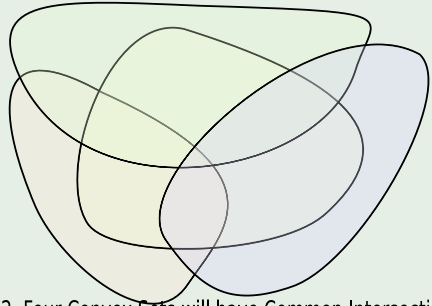


d = 2, Four Convex Sets without Common Intersection, Every two of Which Intersect

Bound $d + 1$ is tight – II

Example

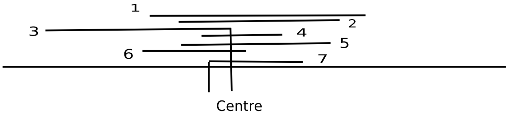
- Same Example with $d = 2$ but with tight bound.



d = 2, Four Convex Sets will have Common Intersection,
Every three of Which Intersect

Bound $d + 1$ is tight (Simple Case) – III

- Case of intervals in 1D.



Helly's Theorem for Intervals

- Centre is between right-most left end-point and left-most right end-point.

Proof of Helly's Theorem

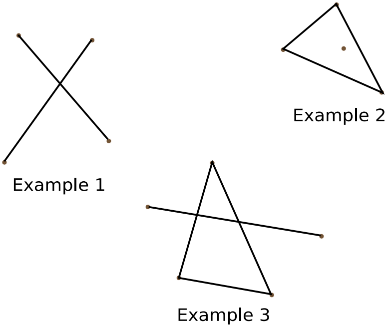
- We need the help of Radon's Theorem to prove Helly's Theorem.
- Actually Radon's and Helly's Theorem are equivalent.

Radon's Theorem

Theorem (Radon's Theorem)

Let P be set of $n \geq d+2$ points in E^d. There exists a partition of P into sets P_1 and P_2 such that convex hulls of P_1 and P_2 intersect (See [2]).

What is the Meaning of Radon's Theorem



Example

Example Instances of Radon's Theorem in E^2.

Sketch of the Proof of Radon's Theorem

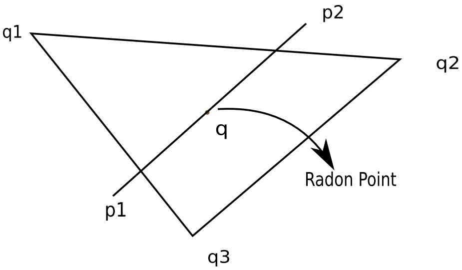
Proof.

- $n \geq d+2$ points are affinely dependent. $\sum_{i=1}^{n} \lambda_i p_i = O, \sum_{i=1}^{n} \lambda_i = 0$, O is origin, not all $\lambda_i = 0$.
- Let I_1 be the set of i for which $\lambda_i > 0$ and I_2 be the set of i for which $\lambda_i < 0$.
- $q_1 = \frac{1}{\lambda} \sum_{i \in I_1} \lambda_i p_i = -\frac{1}{\lambda} \sum_{i \in I_2} \lambda_i p_i = q_2$ where $\lambda = \sum_{i \in I_1} \lambda_i = -\sum_{i \in I_2} \lambda_i$
- q_1 is in the convex hull of points $p_i, i \in I_1$ and q_2 is in the convex hull of points $p_i, i \in I_2$.
- Hence proved (See [2]).

$\square$

Explanation of Proof

Case of five points in 2D.



- $\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{1}{3}q_1 - \frac{1}{3}q_2 - \frac{1}{3}q_3 = 0$
- $q = \frac{1}{2}p_1 + \frac{1}{2}p_1 = \frac{1}{3}q_1 + \frac{1}{3}q_2 + \frac{1}{3}q_3$
- q is the Radon point.

Proof of Helly's Theorem

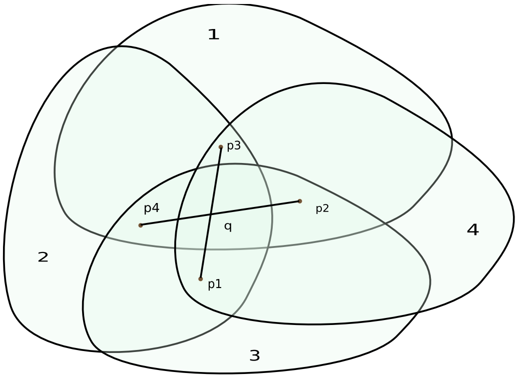
Proof.

- The proof is by mathematical induction.
- Let $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_N$ have a common point p_i (by induction hypothesis).
- Consider P, set of p_i's, which by Radon's theorem cab be partitioned in two sets P_1 and P_2, convex hull of which intersect at q.
- We can prove q belongs to every S_i. If $p_i \in P_1$ then since q is also in convex hull of p_j's in P_2, and also since all $p_j \in S_i$, therefore $q \in S_i$.
- Hence proved (See [2]).

$\square$

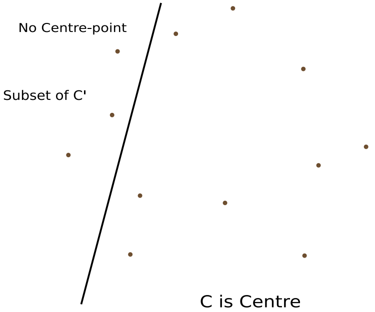
Explanation of Proof

Case of four sets in 2D.



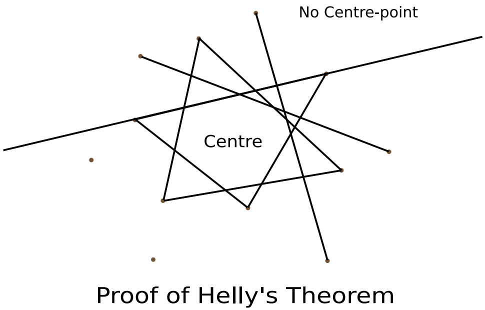
- p_2 and p_4 (also p_3) are contained in S_1.
- q is in the convex hull of p_2 and p_3.
- q is in S_1 because S_1 is convex.
- q is the common intersection point

Observation One for Centre-points



- A half-space containing less than $\lceil \frac{n}{d+1} \rceil$ points will not contain a centre-point.

Observation two for Centre-points



- Centre is intersection of all half-spaces containing more than $\lfloor \frac{nd}{d+1} \rfloor$ points.

Proof of Centre-point

Proof.

- Every $d+1$ half-spaces containing less than $\lceil \frac{n}{d+1} \rceil$ points will not cover all points.
- The intersection of their complements is non-empty.
- By Helly's theorem the intersection of all such complements is non-empty.
- Any point in this intersection satisfies the definition of centre-point.

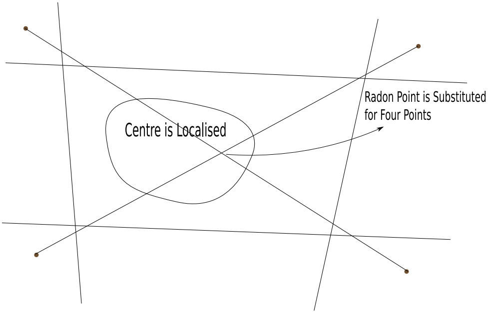
$\square$

Computation of Centre-Point

- Centre-point is in the intersection of all half-spaces containing more than $\lfloor \frac{dn}{d+1} \rfloor$ points.
- Implies an $O({}^nC_d \cdot n)$ algorithm to compute a centre-point.
- Can we do better?

Computation of Centre-Point in Plane

- Centre-point in two dimension can be computed in linear time using Radon's Theorem cleverly (See [4]).



Computation of Approximation Centre-Point

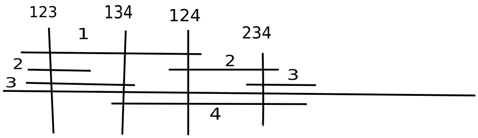
- Approximate centre-point in any dimension can be computed in linear time (See [1]).

Checking Centre-Point

- Problem of checking if a point is a centre-point in linear time in any dimension other than 1D is still not solved. In E^3 it can be done in $O(n^2)$, in E^4 the fastest algorithm needs $O(n^4)$ and for higher dimensions there is only straight-forward method of computing all simplices which is $O(n^{d+1})$.

A Generalisation of Helly's Theorem

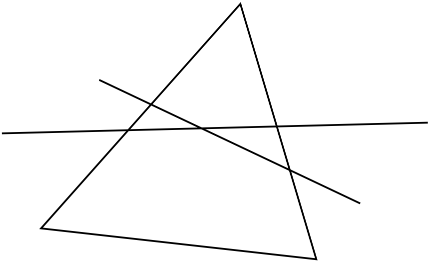
- Let A of size at least $j(d+1)$ be a finite subfamily of K_j^d, the family of all sets of E^d that are the unions of j or fewer pairwise disjoint closed convex set, such that the intersection of every j members of A is also in K_j^d. If every $j(d+1)$ members of A have a point in common, then there is a point common to all the members of A.



Showing That j(d+1) is Tight

Tverberg's Theorem — A Generalisation of Radon's Theorem

- Each set of $(r-1)(d+1)+1$ or more points in E^d can be partitioned into r subsets whose convex hulls have a point in common.



Tverberg's Theorem for Seven Points

A Generalisation of the Centre-point Theorem

Theorem

Given a set P of n points in the plane, there exists two points, not necessarily among input points, that hit all convex sets containing more than $\frac{4}{7}n$ points of P (See [6]).

Summary

- We saw the Helly's Theorem
- Next we proved existence of a Centre-point
- Lastly, we sketched the computation of a Centre-point

References I

K. L. Clarkson, D. Eppstein, G. L. Miller, C. Sturtivant, and S.-H. Teng.
Approximating center points with iterative radon points.
Int. J. Comput. Geom. Appl., 6(3):357–377, 1996.

Herbert Edelsbrunner.
Algorithms in Combinatorial Geometry.
Springer-Verlag, New York, 1987.

Jacob E. Goodman and Joseph O'Rourke, editors.
Handbook of Discrete and Computational Geometry.
Chapman & Hall/CRC, second edition, 2004.

References II

S. Jadhav and A. Mukhopadhyay.
Computing a centerpoint of a finite planar set of points in linear time.
Discrete Comput. Geom., 12:291–312, 1994.

J. Matousek.
Lectures in Discrete Geometry.
Springer-Verlag, New York, 2002.

Nabil H Mustafa and Saurabh Ray.
An optimal extension of the centerpoint theorem.
Computational Geometry, 2007.
doi:10.1016/j.comgeo/2007.10.004.

References III

J. Pach and P. K. Agrawal.
Computational Geometry.
John Wiley & Sons, New York, 1995.

Thank You