Helly's Theorem and Centre Point

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Outline of the Talk

- Motivation
- Helly's Theorem
- Proof of Existence of Centre-point
- Computing Centre-point
- Summary

Helly's theorem (1923) and Radon's theorem (1921) are equivalent and belong to the most fundamental among the non-trivial results in geometry. The concept of centre-point is a generalisation of the concept of median for the points. For a set of points, it is a point such that the partitions of the set defined by it are reasonably balanced.

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Helly's theorem is used to prove the existence of centre-points where as Radon's theorem is used to compute the centre-points in plane in linear time complexity. In three dimensions and above there exists an algorithm that yields an approximate centre-point in linear time that uses Radon's theorem extensively. There are analogous results to Helly's and Radon's for generalisations of configurations and arrangements such as circular sequences and pseudo-line arrangements.

In the talk we shall prove the Helly's theorem, use it to prove the existence of centre-point, and present some algorithms to compute centre-points and to check if a point is centre-point.

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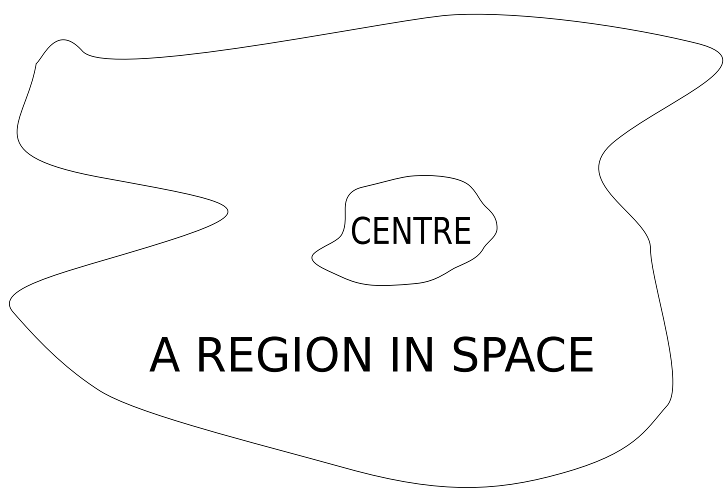
Motivation for Centre Points

- We all have a notion of centre.



- Can we formalise the notion of centre?

We can think of centre of almost every geometric object. We perceive that there should be a central localtion for everything.

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Various Notions of Centre

- Like circumcentre of a triangle — minimise the maximum distance to all points.

- Like orthocentre/in-centre of a triangle — maximise the minimum distance to the exterior.

Both these generalise badly. We need to generalise the notion of median of n values.

Like there is a central place in China which is farthest from the sea in any direction. Its distance to Indian Ocean, Pacific Ocean and Arctic Ocean is same.

In 1-dimension both the notions generalise to the mid-point of the two exterior point. Hence these notions are not at all suitable. Even average of n values captures more information than these.

By the way, we usually use average because it is easier to compute, but one or two very extreme values can have adverse effect on average.

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Property We are Looking for in Centre

- If possible, equal count/area/volume on *all sides*

- Otherwise, balancing these as much as possible

1. What do we mean by a *side* of any point? It is an ambiguous term.

2. The second point is interesting because as we will see it is not always possible to find centre which balances perfectly.

3. We can extend the result to non-euclidean gemotry too, if a k-flat in $k+1$-flat partitions it into two (actually three).

4. The notion of centre that we are looking for does not make sense in spherical (Riemannian) geometry where points are poles. All poles are in one side of a line (there aren't any sides).

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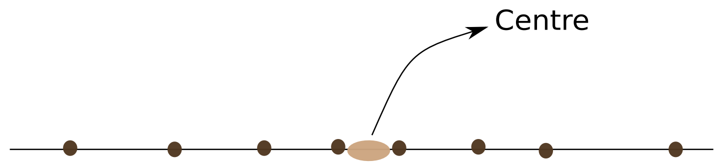
Median is Centre in 1D



Points in a Line

1. We are not interested in distances — But only in number of points in either side.

   So, we do not worry about distance metric.

2. Median partitions the set in two equal sized halves.

The geometers do not like terms 1D, 2D, instead they prefer E^d or R^d to mention the d-dimensional space.

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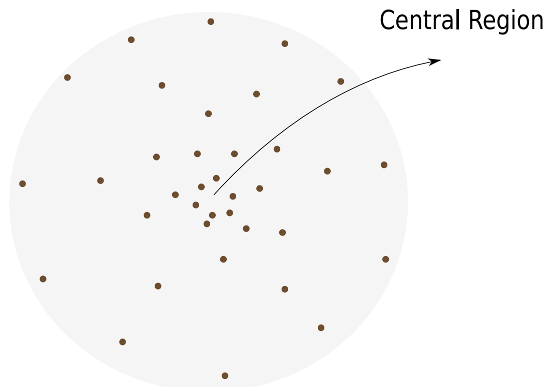
Centre of Uniformly Distributed points in 2D



Uniformly Distributed Points

Here centre is very nicely situated in the middle.

Is it always possible to find a *balanced* centre?

When we say that the centre is nicely situated in the middle, we actually mean that every line passing through it roughly divides the set in half.

The same result will be had if the points are distributed uniformly in a circle. Anything that has to do with circle, we hope to have a good centre.

The word *balanced* is not a technical term. I use it very loosely because we will see later that we will need to stretch the meaning of this term quite a bit.

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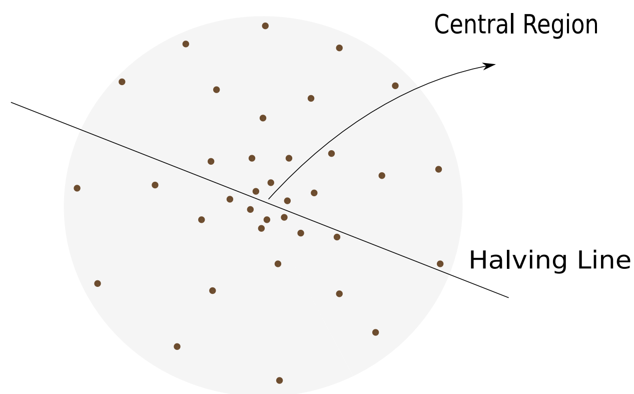
What do We mean by *Balanced* Centre



Uniformly Distributed Points

Every line through centre divides the set in half.

So, is it possible to find a *balanced* centre in all cases?

In 2D it is always possible to find median of n points that partitions in half. But it is not so in higher dimensions.

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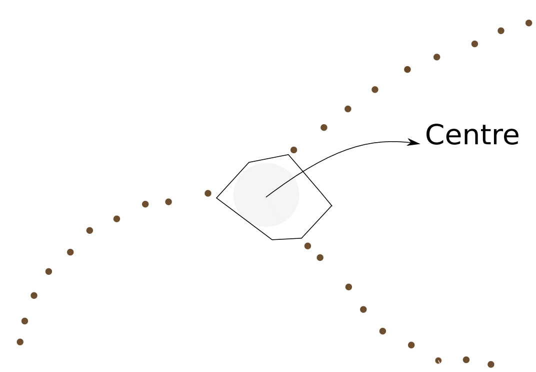
Worst Case Distribution in 2D

- Median divides points in equal halves.

- But in 2D, the answer is NO.



A Difficult Set of Points

- Centre point dividing points only in one-third.

Here we can only hope to find a centre-point which guarantees only one-third of points in either sides.

In fact, here I have pointed out the centre according to the formal definition that we will give later.

What happens in still higher dimensions?

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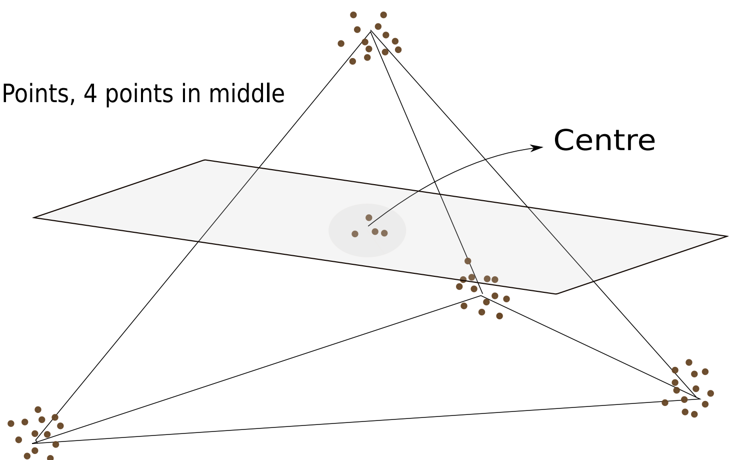
Worst Case Distribution in 3D



Another Difficult Distribution of Points in 3D

Centre point *dividing* points only in one-fourth.

This is another kind of worst case distribution, but in 3D. The idea is similar, you put a group of point on each vertex of a simplex and put a few points somewhere in the middle.

Here, when we say that the centre *divides* the set of points in one-forth, what we really mean is that there exist a plane passing through the centre which contains only one-fourth of the points on one side.

Again *dividing* is taken in a loose sense.

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Definition of a Centre-point

A *centre-point* of a finite set of points, P, is a point such that every closed half-space containing it contains at least $\lceil \frac{n}{d+1} \rceil$ points of P.

Equivalently, every hyper-plane passing through the centre-point contains at least $\lceil \frac{n}{d+1} \rceil$ points.

Mostly we will be looking at the discrete and finite set of points. So we will be interested only in the count. But it is possible to generalise for other objects, if we try to balance area, volume, etc.

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Remarks on the Definition of Centre-point

We will concentrate only in E^d, even though centre-points exists elsewhere.

What is important is the notion of half-space. Not all geometrical spaces may have this notion.

We have already pointed out the case of non-euclidean spherical geometry case where points are poles and lines are equatorial chords.

It is required that every hyperplane should create two half-spaces. So essentially it divides the space in three sets, two half-spaces and third the points in the hyperplane itself.

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Definition of a Centre-point in Lower Dimensions

1. In 1D — Centre-point is a median point partitioning set in half.

2. In 2D — Centre-point partitions set in one-third at least.

3. In 3D — Centre-point partitions set in one-fourth at least.

The *at least* clause in 2D and 3D is important because there are better cases when centre-point *divides* the set more equally.

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Questions We can Raise

- Do centre-points always exist?

- Are they unique?

Since we have already seen that in higher dimensions than 1 there may not exist any point that divides the set in half by *every* hyperplane passing through it; it is natural to ask whether or not same happens for $\lceil \frac{n}{d+1} \rceil$.

For the second question, we may safely say that they may not be unique.

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The Answers

- Do centre-points always exist?

  Yes. However, we need to supply a proof.

- Are they unique?

  No. The examples above are sufficient to show that there may be many centre-points, some better and some worse.

We know that medians are not unique. And for a set of even number of points, there are many points between two middle points which partition the set in equal halves.

As we pointed out in case of better distributed point centre-points can do better.

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Task Before Us

How do we prove the existence of centre-point (at least in E^d) according to the definition given previously?

Hopefully, the proof may generalise easily to other spaces which admit concrete notions of half-spaces.

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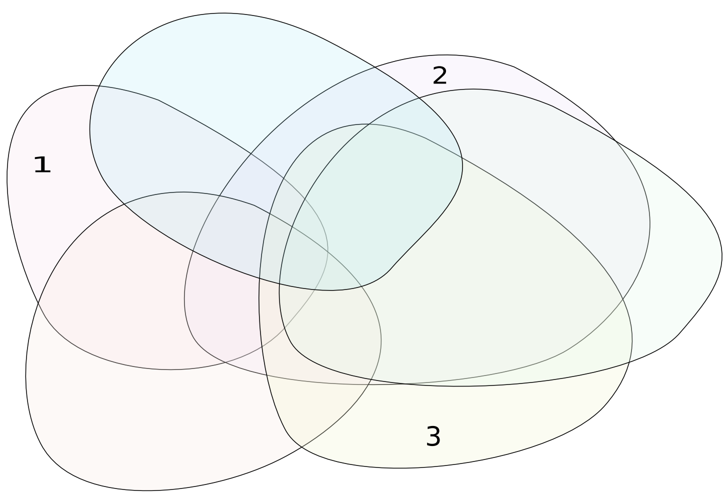
Helly's Theorem

We need to take help of Helly's theorem to prove the existence of centre point.



Six Convex Sets, Every Three of Which are Intersecting

Helly's Theorem Let S_1, S_2, ..., S_n be $n \geq d+1$ convex sets in E^d. If every $d+1$ of the sets have common intersection than all the sets have a common intersection.

That is to say that $\bigcap_{i=1}^{n} S_i$ is non-empty.

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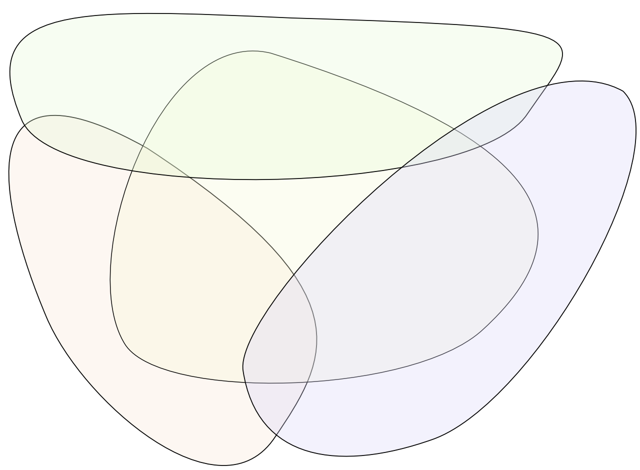
Bound $d + 1$ is tight $-$ I

- Counter Example in $d = 2$ when we guarantee only every d sets have common intersection.



d = 2, Four Convex Sets without Common Intersection, Every two of Which Intersect

The convex sets on the outside do not intersect.

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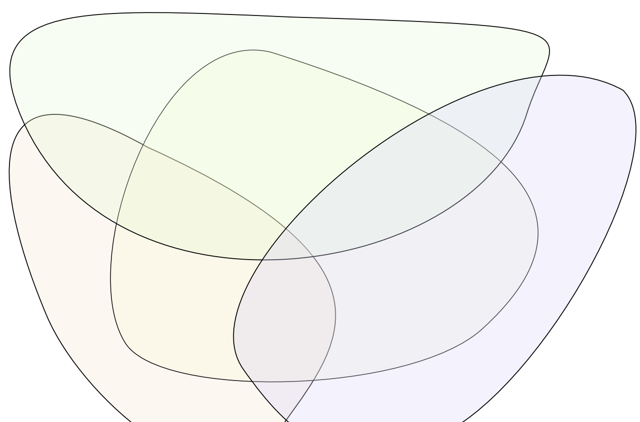
Bound $d + 1$ is tight $-$ II

- Same Example with $d = 2$ but with tight bound.



d = 2, Four Convex Sets will have Common Intersection,
Every three of Which Intersect

You can check the case in 1 dimension, if every two intervals of several given intervals intersect then all of them have a common intersection.

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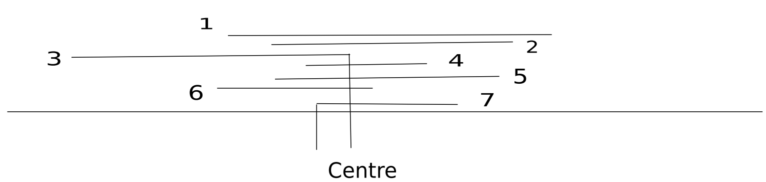
Bound $d + 1$ is tight (Simple Case) – III

- Case of intervals in 1D.



Helly's Theore for Intervals

- Centre is between right-most left end-point and left-most right end-point.

It is easy to prove that points that are in the central region shown in slide are in every interval. Otherwise there will exist two intervals which do not intersect.

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Proof of Helly's Theorem

- We need the help of Radon's Theorem to prove Helly's Theorem.

- Actually Radon's and Helly's Theorem are equivalent.

That is to say that Radon's theorem can be used to prove Helly's theorem and vice-versa.

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Radon's Theorem

Radon's Theorem Let P be set of $n \geq d + 2$ points in E^d. There exists a partition of P into sets P_1 and P_2 such that convex hulls of P_1 and P_2 intersect.

Radon's theorem is a classic result from geometry.

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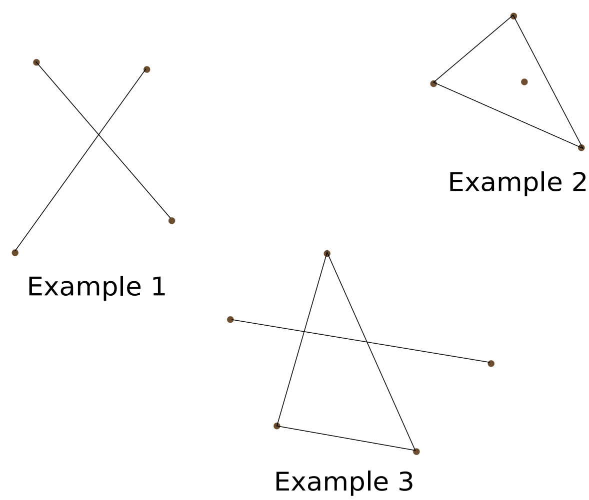
What is the Meaning of Radon's Theorem



Example Instances of Radon's Theorem in E^2.

We can give examples in 3D also. But, it is better to prove the result once for all.

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Sketch of the Proof of Radon's Theorem

- $n \geq d + 2$ points are affinely dependent.

  $\sum_{i=1}^{n} \lambda_i p_i = O, \sum_{i=1}^{n} \lambda_i = 0$, O is origin, not all $\lambda_i = 0$.

- Let I_1 be the set of i for which $\lambda_i > 0$ and I_2 be the set of i for which $\lambda_i < 0$.

- $q_1 = \frac{1}{\lambda} \sum_{i \in I_1} \lambda_i p_i = -\frac{1}{\lambda} \sum_{i \in I_2} \lambda_i p_i = q_2$ where $\lambda = \sum_{i \in I_1} \lambda_i = -\sum_{i \in I_2} \lambda_i$

- q_1 is in the convex hull of points $p_i, i \in I_1$ and q_2 is in the convex hull of points $p_i, i \in I_2$.

- Hence proved.

It is enough if we appreciate that we partition depending on whether λ_i is positive or not.

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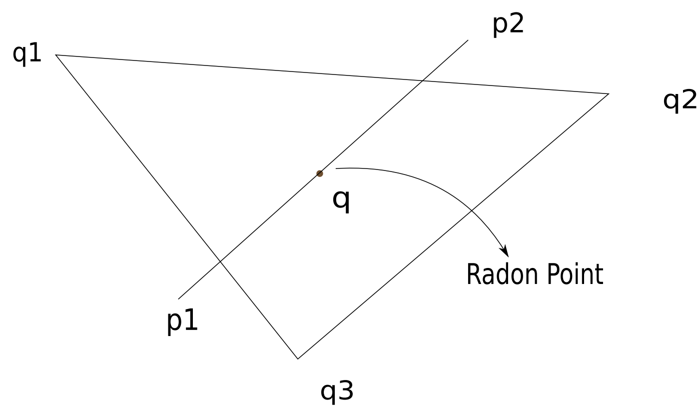
Explanation of Proof

Case of five points in 2D.



- $\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{1}{3}q_1 - \frac{1}{3}q_2 - \frac{1}{3}q_3 = 0$

- $q = \frac{1}{2}p_1 + \frac{1}{2}p_1 = \frac{1}{3}q_1 + \frac{1}{3}q_2 + \frac{1}{3}q_3$

- q is the Radon point.

p_i's are the points with positive λ's and q_i's are the points with negative λ's.

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Proof of Helly's Theorem

- The proof is by mathematical induction.

- Let $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_N$ have a common point p_i (by induction hypothesis).

- Consider P, set of p_i's, which by Radon's theorem cab be partitioned in two sets P_1 and P_2, convex hull of which intersect at q.

- We can prove q belongs to every S_i. If $p_i \in P_1$ then since q is also in convex hull of p_j's in P_2, and also since all $p_j \in S_i$, therefore $q \in S_i$.

- Hence proved.

Yet another proof, this time for Helly's Theorem

If $p_i \in P_1$ then since q is in convex hull of p_j's in P_2 implies $p_j \in S_i$. Every point in other set is in S_i.

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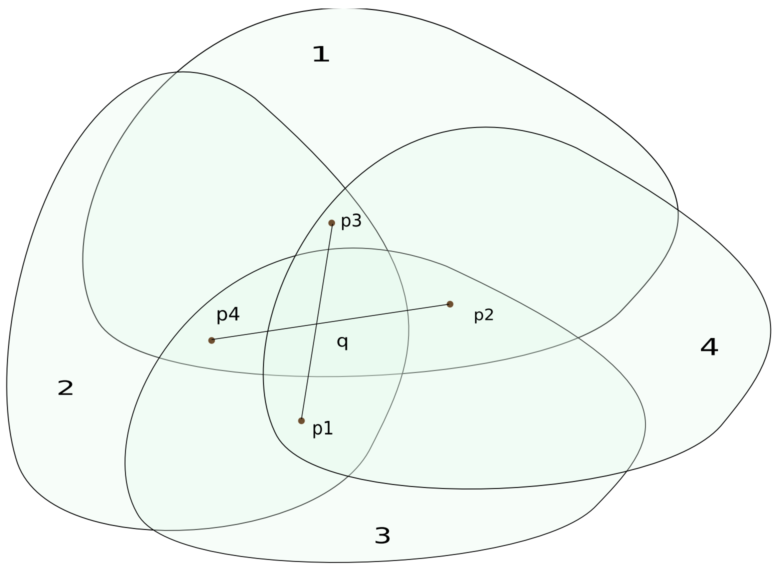
Explanation of Proof

Case of four sets in 2D.



- p_2 and p_4 (also p_3) are contained in S_1.
- q is in the convex hull of p_2 and p_3.
- q is in S_1 because S_1 is convex.
- q is the common intersection point.

Similar to S_1 we can prove that q belongs to all S_i.

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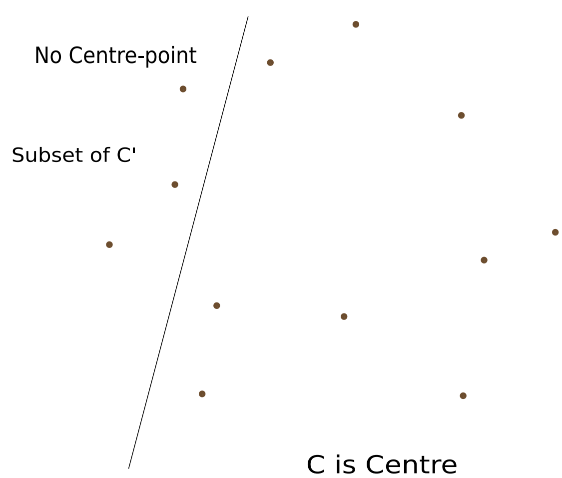
Observation One for Centre-points



- A half-space containing less than $\lceil \frac{n}{d+1} \rceil$ points will not contain a centre-point.

We can prove this by contradiction. If it contains a centre-point, we may find a half-space parallel to the half-space in question which contains even lesser points.

Or otherwise also, it is contained in a half-space which contains less than $\lceil\frac{n}{d+1}$ points, and therfore will not be a centre-point according to the definition that we have given.

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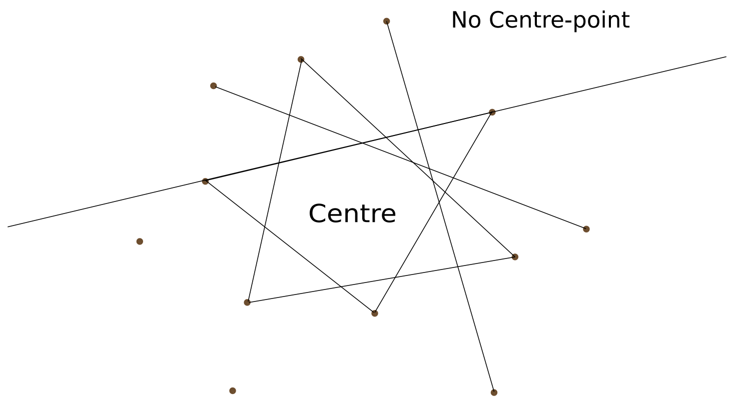
Observation two for Centre-points



Proof of Helly's Theorem

- Centre is intersection of all half-spaces containing more than $\lfloor \frac{nd}{d+1} \rfloor$ points.

Now we can take union of all such half spaces which do not contain at lease one centre-point.

If we take complement of this set then we get the set of all centre-points.

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Proof of Centre Point

- Every $d+1$ half-spaces containing less than $\lceil \frac{n}{d+1} \rceil$ points will not cover all points.

- The intersection of their complements is non-empty.

- By Helly's theorem the intersection of all such complements is non-empty.

- Any point in this intersection satisfies the definition of centre-point.

Centre can be equivalently seen as the intersection of all half-spaces containing more than $\lfloor \frac{dn}{d+1} \rfloor$ points.

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Computation of Centre-Point

- Centre-point is in the intersection of all half-spaces containing more than $\lfloor \frac{dn}{d+1} \rfloor$ points.
- Implies an $O({}^{n}C_{d} \cdot n)$ algorithm to compute a centre-point.
- Can we do better?

We can do better at least for plane. For higher dimensions, there is a good chance that there is a linear time algorithm. But right now, we cannot do better than $O(n^{d+1})$.

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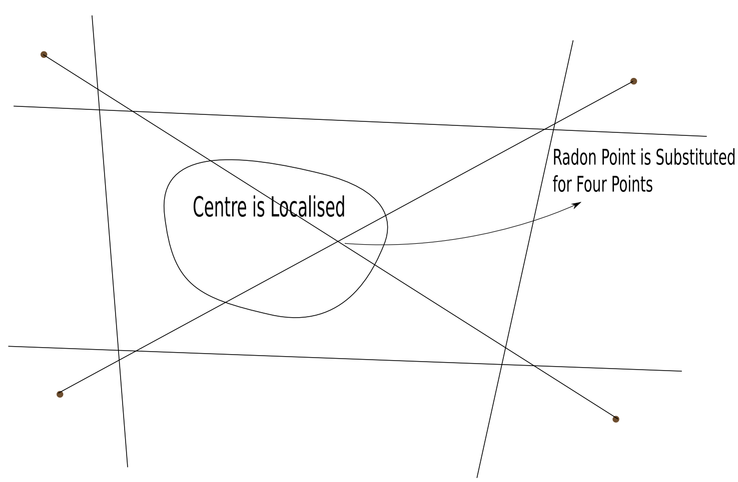
Computation of Centre-Point in Plane

- Centre-point in two dimension can be computed in linear time using Radon's Theorem cleverly.



This algorithm is analogous to the algorithm to find median of a set of points in plane. It is a modification of prune and search class of algorithms.

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Computation of Approximation Centre-Point

- Approximate centre-point in any dimension can be computed in linear time.

The algorithm repeatedly substitutes a small set of few extreme points with their Radon point.

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Checking Centre-Point

- Problem of checking if a point is a centre-point in linear time in any dimension other than 1D is still not solved.

  In E^3 it can be done in $O(n^2)$, in E^4 the fastest algorithm needs $O(n^4)$ and for higher dimensions there is only straight-forward method of computing all simplices which is $O(n^{d+1})$.

More or less we have to construct every combination of half-plane passing through the centrepoint in question. Thus the complexity of the algorithm is similar to the construction of Centre.

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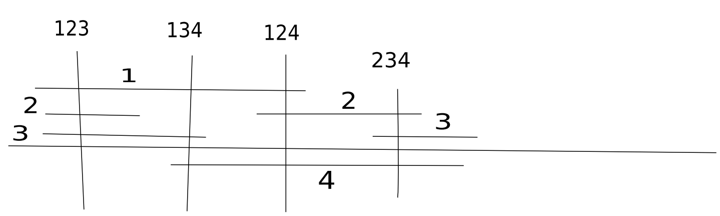
A Generalisation of Helly's Theorem

- Let A of size at least $j(d+1)$ be a finite subfamily of K_j^d, the family of all sets of E^d that are the unions of j or fewer pairwise disjoint closed convex set, such that the intersection of every j members of A is also in K_j^d. If every $j(d+1)$ members of A have a point in common, then there is a point common to all the members of A.



Showing That j(d+1) is Tight

Bound $j(d+1)$ is tight and cannot be further reduced.

There is no common intersection for set of two intervals if every three of them intersect commonly.

We are relaxing the condition that each set needs to be convex. Instead we say that each set is disjoint union of j or less number of convex sets.

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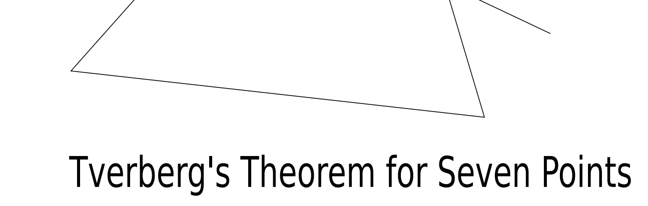
Tverberg's Theorem — A Generalisation of Radon's Theorem

- Each set of $(r-1)(d+1)+1$ or more points in E^d can be partitioned into r subsets whose convex hulls have a point in common.



Tverberg's Theorem for Seven Points

Bound $(r-1)(d+1)+1$ is tight and cannot be further reduced.

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Summary

- We saw the Helly's Theorem

- Next we proved existence of a Centre-point

- Lastly, we sketched the computation of a Centre-point

Further Reading

- Jacob E. Goodman and Joseph O'Rourke (eds.), *Handbook of Discrete and Computational Geometry*, 2nd edition, 2004

- Herbert Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer Verlag, 1987

Thank You