

Geometric Graphs

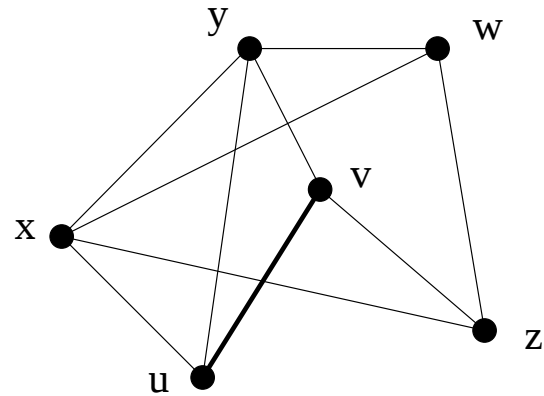
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Workshop on Introduction to Graph and Geometric Algorithms

PSG College of Technology, Coimbatore

Geometric Graph



- ★ $V =$ set of geometric objects (point set in the plane)
- ★ $E = \{(u, v)\}$ based on some geometric condition

Questions on Geometric Graphs

- ★ Problems on graphs
 - ✿ Independent set, coloring, clique, etc.

- ★ Combinatorial/Structural questions
 - ✿ Obtain **Bounds**
 - ✿ Characterization

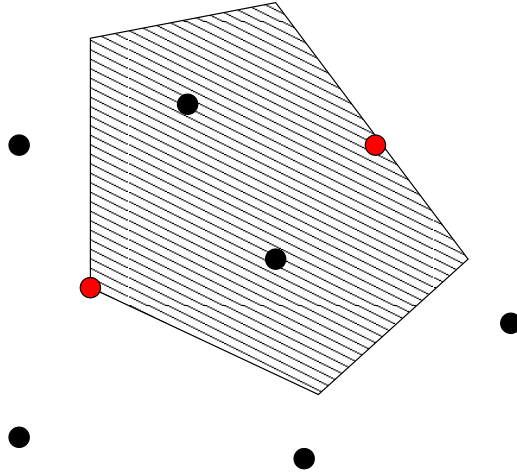
- ★ Computational questions
 - ✿ Efficient Algorithm
 - ✿ Approximation

Geometric graphs

- ☆ V - set of geometric objects
- ☆ E - object i and j satisfy certain geometric condition
- ☆ Broad classes of geometric graphs (based on edge condition)
 - ✿ Proximity graphs
 - ✿ Intersection graphs
 - ✿ Distance based graphs

Proximity Graphs

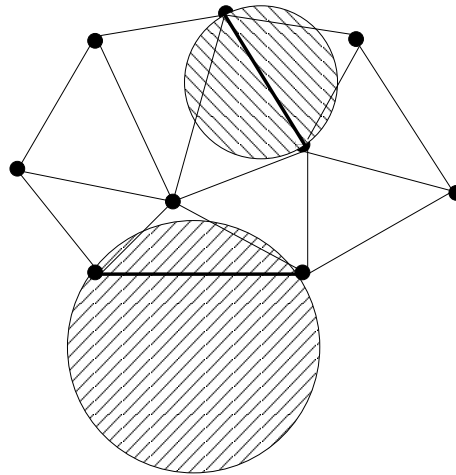
- ☆ P - point set in plane
- ☆ $R_{i,j}$ - proximity region defined by i and j



- ☆ V - point set P
- ☆ $(i, j) \in E$ if $R_{i,j}$ is empty
- ☆ Examples - Delaunay, Gabriel, Relative Neighborhood Graph
- ☆ Applications - Graphics, wireless networks, GIS, computer vision, etc.

Delaunay Graph - Classic Example

★ P - point set in plane



★ V - point set P

★ $(i, j) \in E$ if $\exists$ some empty circle thro' i and j

★ Triangle (i, j, k) if $\text{circumcircle}(i, j, k)$ is empty
(Equivalent condition)

★ Applications - Graphics, mesh generation, computer vision, etc.

Questions on Delaunay Graph

★ Combinatorial - Bounds on

- ✿ Maximum size of edge set?

- ✿ Chromatic number?

- ✿ Maximum independent set?

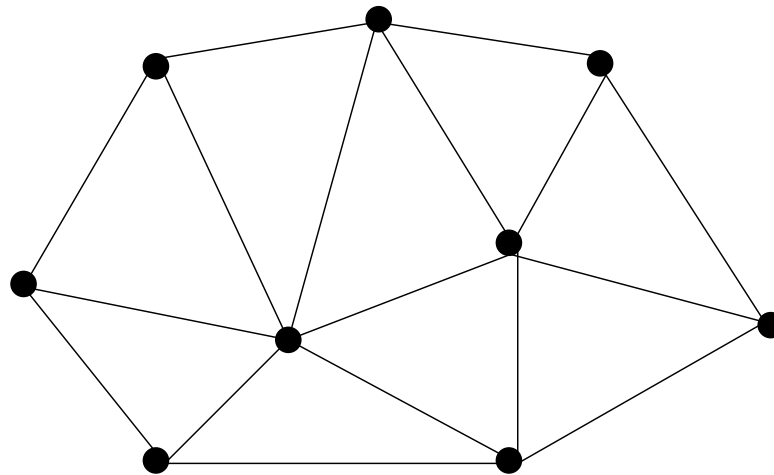
(Over all possible point sets P)

★ Computational

- ✿ Efficient Algorithm

Delaunay Graph - Classic Example

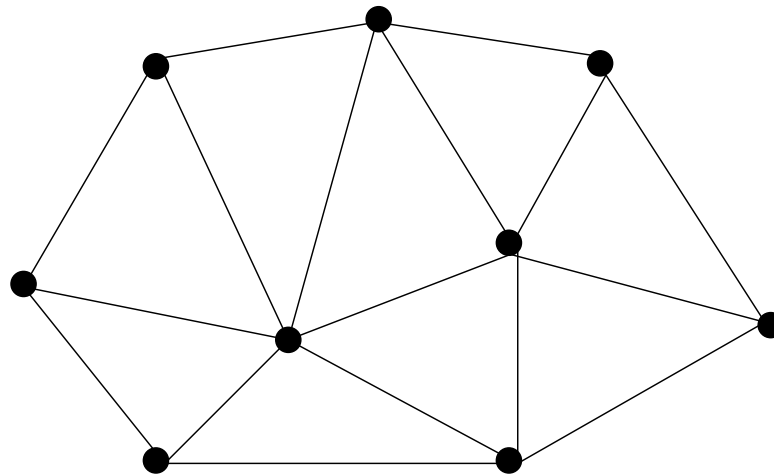
★ P - point set in plane



★ Observations:

Delaunay Graph - Classic Example

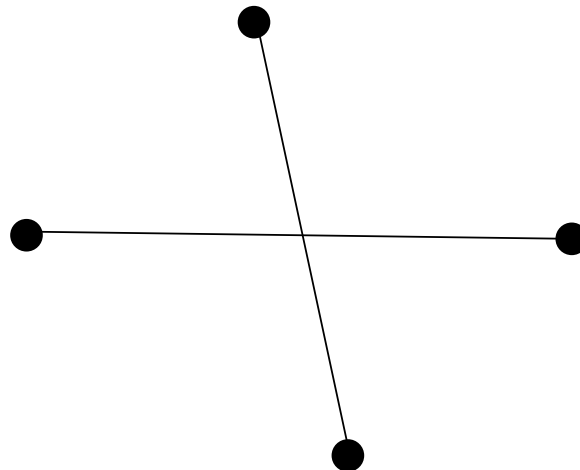
★ P - point set in plane



★ Observations: Planar?

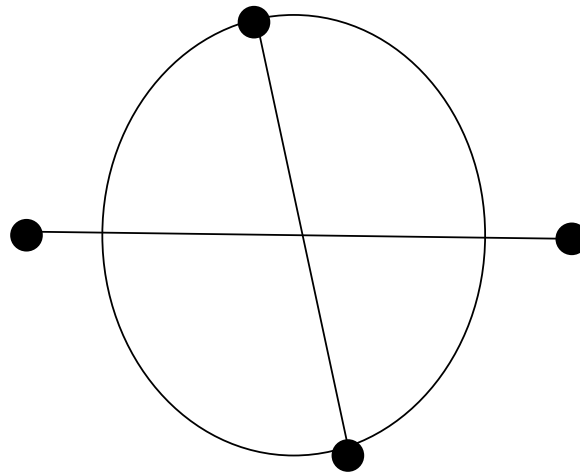
Delaunay Graph - Planar

☆ Let, if possible, 2 edges **CROSS**



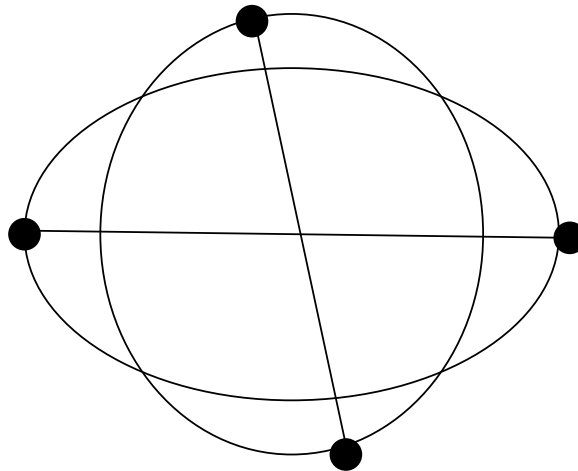
Delaunay Graph - Planar

☆ Let, if possible, 2 edges **CROSS**



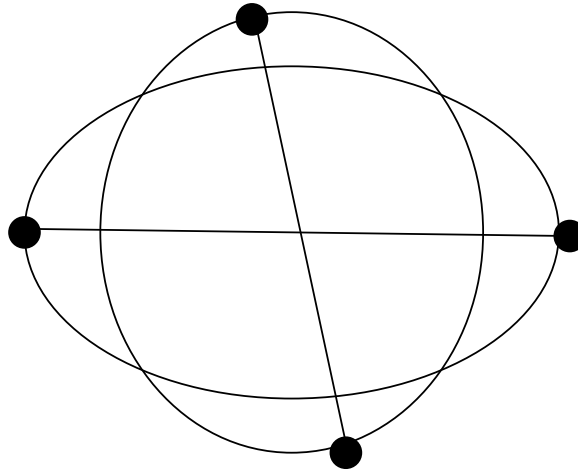
Delaunay Graph - Planar

☆ Let, if possible, 2 edges **CROSS**



Delaunay Graph - Planar

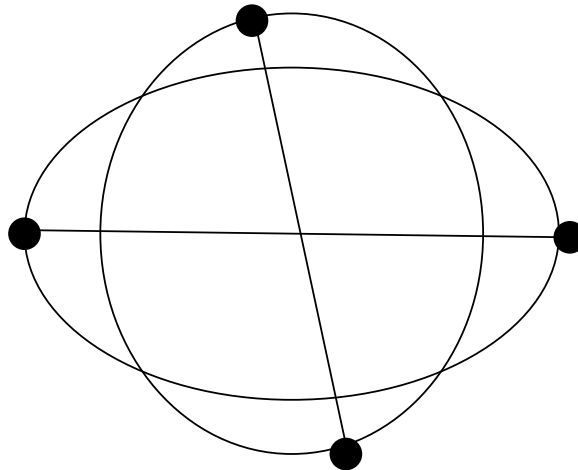
☆ Let, if possible, 2 edges **cross**



☆ Circles c'ant intersect like this (why?)

Delaunay Graph - Planar

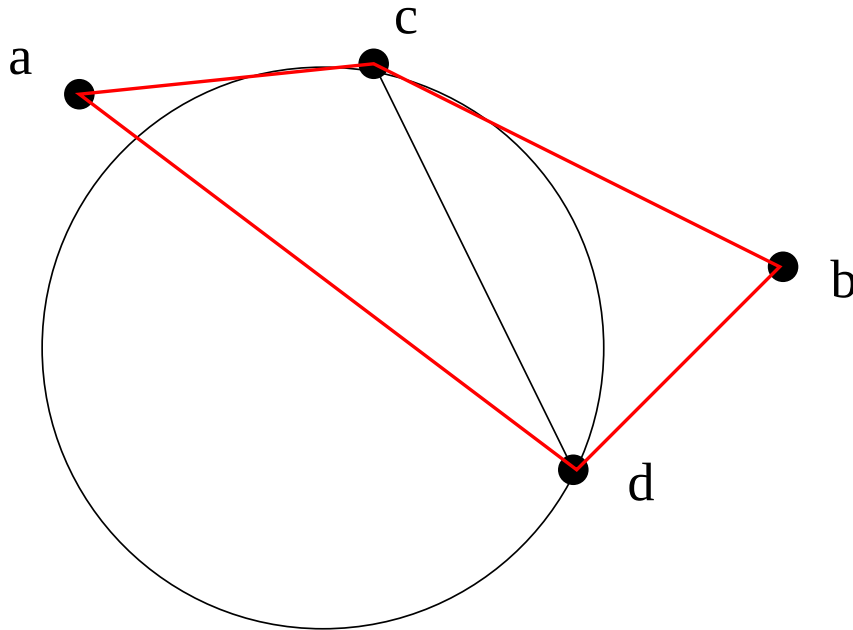
- ☆ Let, if possible, 2 edges **cross**



- ☆ Circles can't intersect like this (why?)
- ☆ One endpoint of an edge lies within the other circle
 - ✿ Contradiction

Delaunay Graph - Proof using angles

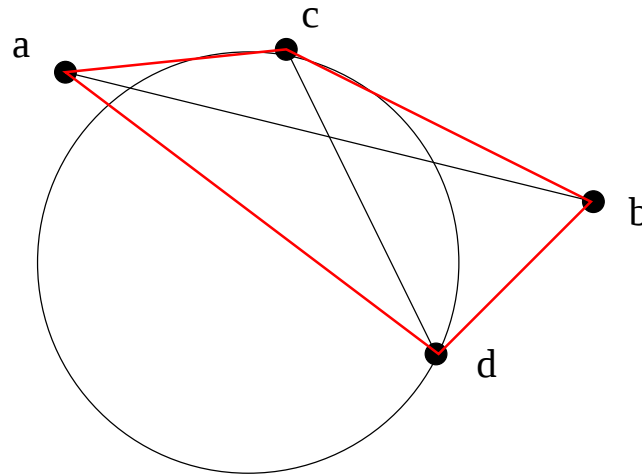
- ☆ Consider any circle passing through c and d
- ☆ Points a and b are outside the circle



☆ $\angle cad + \angle cbd < 180$

Delaunay Graph - Proof using angles

- ☆ Let, if possible, edges ab and cd **cross**
- ☆ Consider the quadrilateral $acdb$



- ☆ cd is an edge $\implies \angle cad + \angle cbd < 180$
- ☆ ab is an edge $\implies \angle acb + \angle adb < 180$
- ☆ $\angle cad + \angle cbd + \angle acb + \angle adb < 360$
- ✿ Contradiction

Questions on Delaunay Graph

★ Given any n -point set P in the plane

✿ Delaunay graph is planar

★ Maximum size of edge set

✿ $\leq 3n - 6$ (Euler's formula)

★ Chromatic number

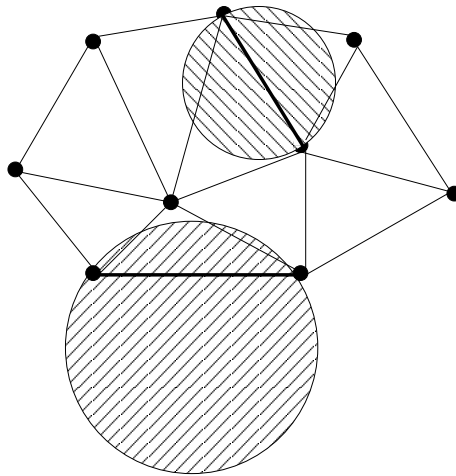
✿ ≤ 4 (Four color theorem)

★ Maximum independent set

✿ $\geq n/4$ (Chromatic number)

Delaunay Graph

★ P - point set in plane

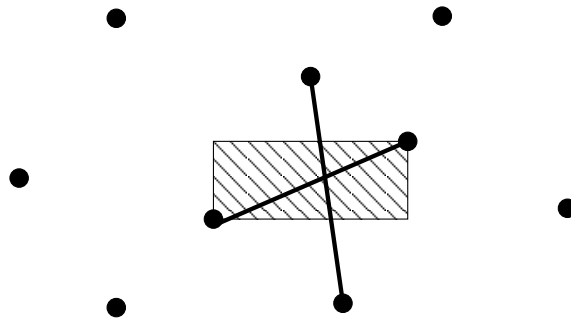


★ V - point set P

★ $(i, j) \in E$ if $\exists$ some empty **circle** thro' i and j

Delaunay Graph - Variants

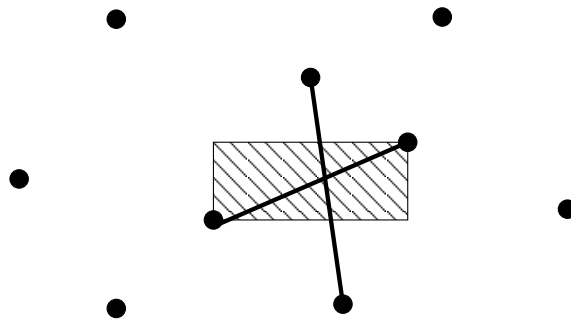
- ★ Edges defined by other objects (instead of circles)
- ★ $(i, j) \in E$ if $\exists$ some empty **rectangle** thro' i and j



- ★ Bounds on the size of maximum independent set?
- ★ Application: Frequency assignment in wireless networks

Delaunay Graph wrt Rectangles

★ $(i, j) \in E$ if $\exists$ some empty rectangle thro' i and j



★ Graph Properties

✿ Graph can have $\Omega(n^2)$ edges

✿ $K_n, n \geq 5$ is a forbidden subgraph

Bounds on Independent Set Size

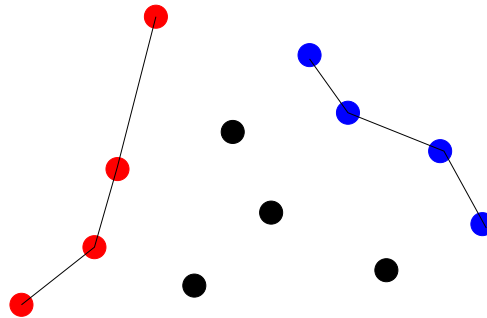
Theorem: Any Delaunay graph (wrt rectangles) has an independent set of size at least $\sqrt{n}/2$

Bounds on Independent Set Size

☆ Same slope sequence of points

⌘ +ve slope sequence (Red)

⌘ -ve slope sequence (Blue)



☆ Same slope sequence of size $2k$

⌘ Independent set of size k

Bounds on Independent Set Size

Erdoes-Szekeres Theorem: Let P be any set of $m^2 + 1$ points in the plane. There exists a same slope sequence (+ve or -ve) of size $m + 1$.

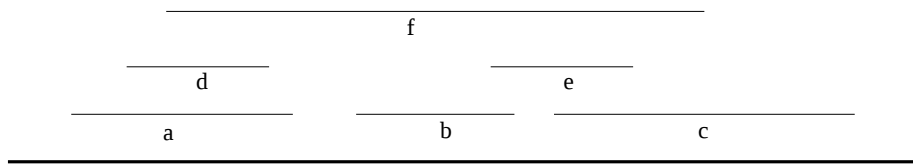
- ★ Atleast six different proofs
(Monotone subsequence survey by Michael Steele)
- ★ Let S be any sequence of $m^2 + 1$ integers. There exists a monotonic subsequence (increasing or decreasing) of size $m + 1$.

Independent Set - Open Problem

- ★ Size of maximum independent set - Lower bound
 - ✿ $\Omega(n^{0.5})$ (Slope sequence)
 - ✿ Improved to $\Omega(n^{0.618-\epsilon})$ (Ajwani et al, SPAA '07)
- ★ Size of maximum independent set - Upper bound
 - ✿ $O(n/\log n)$ (Pach et al '08)
- ★ Conjecture: Close to $O(n/\log n)$
- ★ Open problem : Obtain better upper/lower bounds

Intersection Graphs

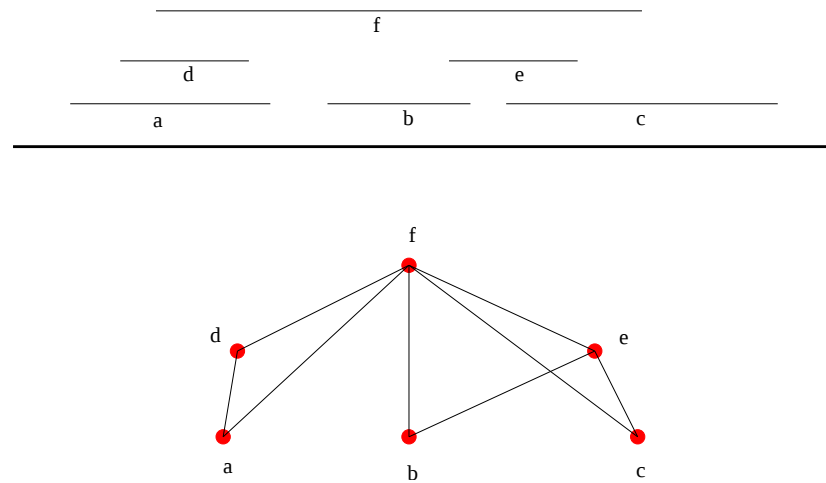
- ★ Interval Graph - Classic example
- ★ S - set of geometric objects s_i (intervals on the real line)



- ★ V - set of object s_i
- ★ $(s_i, s_j) \in E$ if objects s_i and s_j intersect

Interval Graphs

☆ S - set of intervals on the line



☆ V - set of object s_i

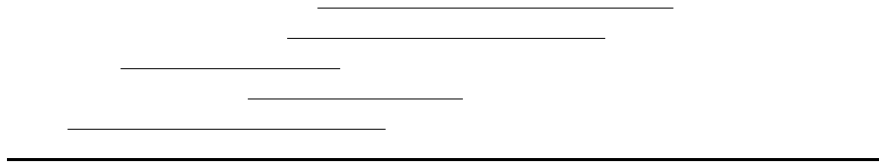
☆ $(s_i, s_j) \in E$ if objects s_i and s_j intersect

☆ Graph problems - Maximum independent set, Maximum clique, Chromatic number, etc.

✿ Can be computed efficiently

Intervals

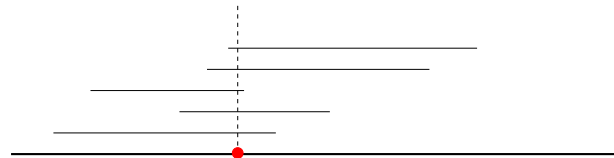
- ★ S - set of intervals on the real line
- ★ Every 2 intervals in S intersect



- ★ Claim: All the intervals have a common intersection

Intervals

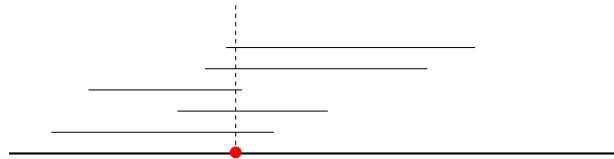
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Intervals

- ★ S - set of intervals on the real line
- ★ Every 2 intervals in S intersect
- ★ Claim: All the intervals have a common intersection



- ★ Induction proof (Exercise)
- ★ Constructive proof
 - 🌀 Construct a point p that is contained in all the intervals

Intervals

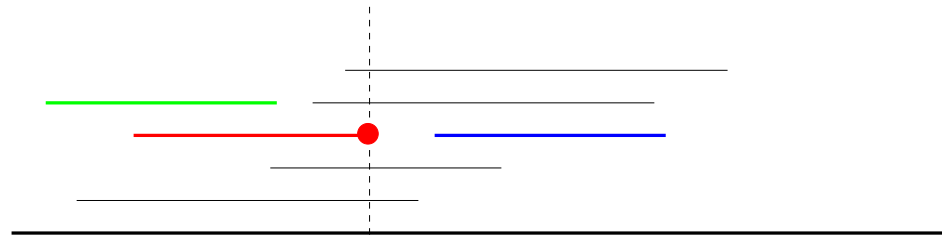
- ☆ S - set of intervals on the real line
- ☆ Every 2 intervals intersect
- ☆ Constructive proof
 - ✿ Construct a point p that is contained in all the intervals
- ☆ p : Right endpoint of interval that ends leftmost
 - ✿ Leftmost right endpoint



- ☆ Claim: All the intervals contain p

Intervals

- ☆ Construct a point p that is contained in all the intervals
- ☆ p : Right endpoint of interval that ends leftmost
 - ✿ Leftmost right endpoint
- ☆ Claim: All the intervals contain p
- ☆ Proof by contradiction



Intervals

- ★ S - set of intervals on the real line
- ★ Every 2 intervals intersect
 - ✿ There exists a point p contained in all the intervals
- ★ Out of every 3 intervals, some 2 of them intersect
 - ✿ Claim: There exists two points p and q such that every interval contains either p or q .
- ★ Out of every r intervals, some s of them intersect, $r \leq s$
 - ✿ Claim: There exists a set of $r - s + 1$ points such that every interval contains one of them

Intersection Graphs of Axis Parallel Rectangles

☆ S - set of axis parallel rectangles

☆ Every 2 rectangles intersect

✿ Claim: There exists a point p contained in all the rectangles

✿ Is it true?

Intersection Graphs of Circles

- ★ S - set of circles
- ★ Every 2 circles intersect
- ✿ Claim: There exists a point p contained in all the circles

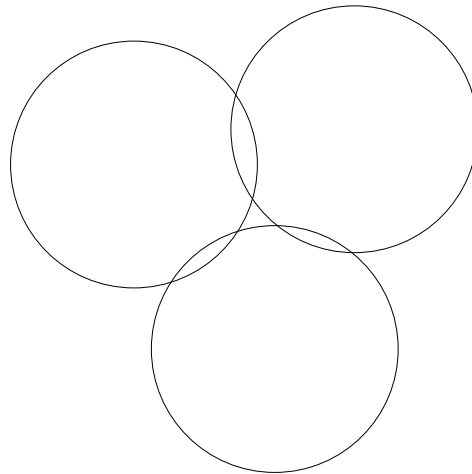
Intersection Graphs of Circles

★ S - set of circles

★ Every 2 circles intersect

✿ Claim: There exists a point p contained in all the circles

✿ Not true



Intersection Graphs of Circles

- ★ S - set of circles

- ★ Every 2 circles intersect

 - ✿ Claim: There exists a point p contained in all the circles

 - ✿ Not true

- ★ Every 3 circles intersect

 - ✿ Claim: There exists a point p contained in all the circles

 - ✿ True

- ★ Helly Theorem: Statement true for convex objects

Intersection Graphs of Circles - Open Problem

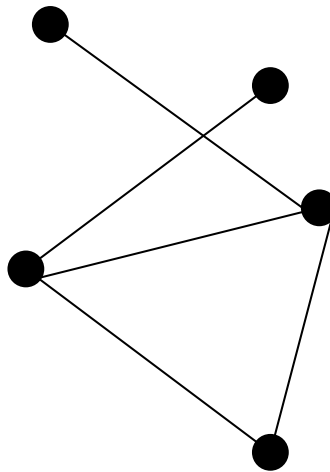
- ☆ S - set of circles
- ☆ Every 3 circles intersect
 - ✿ Claim: There exists a point p contained in all the circles
- ☆ Out of every 4 circles, some 3 of them intersect
- ☆ How many points needed to "hit" all the circles
 - ✿ 3 points are needed
 - ✿ 5 points are sufficient (Gallai)
- ☆ Out of every 4 convex sets, some 3 of them intersect
- ☆ How many points needed to "hit" all the convex sets
 - ✿ 3 points are needed
 - ✿ 13 points are sufficient (Kleitman, Gyarfas, Toth '01)

Distance based Graphs

☆ Unit distance graph

✿ V - point set in plane

✿ $(i, j) \in E$ if $d(i, j) = 1$



☆ Place points so as to maximize the number of edges

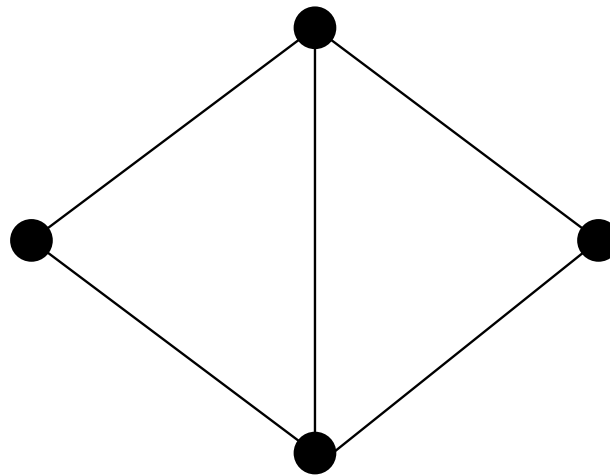
☆ Can you get a complete graph? (even for $n = 4$)

Distance based Graphs

★ Unit distance graph

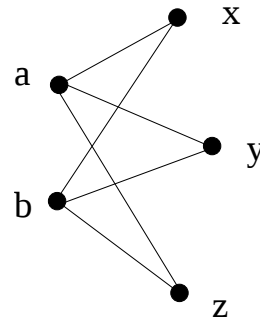
✿ V - point set in plane

✿ $(i, j) \in E$ if $d(i, j) = 1$



Unit Distance Graph

- ☆ V - point set P
- ☆ $(i, j) \in E$ if $d(i, j) = 1$
- ☆ Maximum number of edges? (Erdos)
 - ✿ Over all possible n -point set
- ☆ $O(n^{3/2})$ edges
 - ✿ Forbidden $K_{2,3}$



- ☆ $O(n^{4/3})$ edges
 - ✿ Crossing Lemma, Cuttings, Arrangement of Circles

Unit Distance Graph - Open Problem

☆ Upper bound

✿ $O(n^{4/3})$ edges

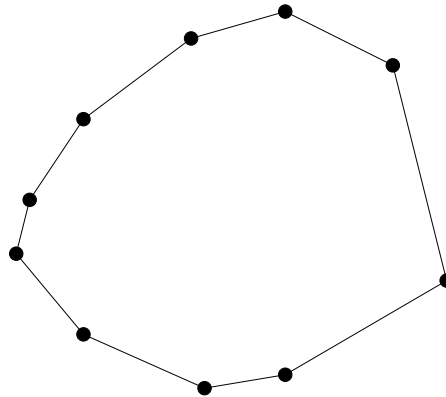
☆ Lower bound

✿ $\Omega(n^{1+\frac{c}{\log \log n}})$ [Erdos]

☆ Conjecture: Lower bound is tight

Unit Distance Graph - Convex Point Set

★ Convex Point Set



★ Upper bound: $O(n \log n)$ edges

★ Lower bound: $2n - 7$ edges

★ Conjecture: Lower bound is tight ($2n$ edges)

Questions

Questions