Graph Partitioning

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Outline

Graph Partitioning

- Partition the vertices and/or edges of a graph.
- Partition must satisfy specified properties.
- Does there exist a partition with specified properties?
- Optimize a specified cost function associated with possible partitions.
- Variety of graph partitioning problems.

Graph Coloring

- Partition the vertex set.
- No two vertices in the same part should be adjacent.
- Number of parts is at most k.
- Does there exist such a partition?
- Minimize the number of parts.
- NP-Hard in general.

Min and Max Cut

- Partition the vertex set.
- Number of parts is 2.
- Minimize (or maximize) number of edges with an end vertex in each part.
- Min-cut can be solved in polynomial-time.
- Max-cut is NP-Hard.

Arborocity

- Partition the edges.
- Each part should be acyclic.
- Minimize the number of parts.
- Solvable in polynomial-time.

Connected Partitions

- Partition the vertices.
- Number of parts and size of each part specified.
- Each part should induce a connected subgraph of the graph.
- Does there exist such a partition?
- NP-Hard in general, even if number of parts is 2.
- Generalization of perfect matchings.

Formal Definition

- Input
  - A graph G with n vertices.
  - Positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$.
- Output
  - A partition $V_1, V_2, \ldots, V_k$ of $V(G)$ such that $|V_i| = n_i$ and V_i induces a connected subgraph of G, if it exists.
- We call such a partition a k-partition of G.

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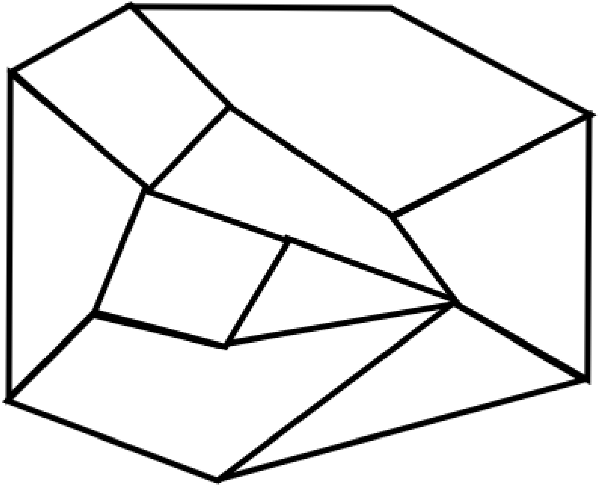
Motivation



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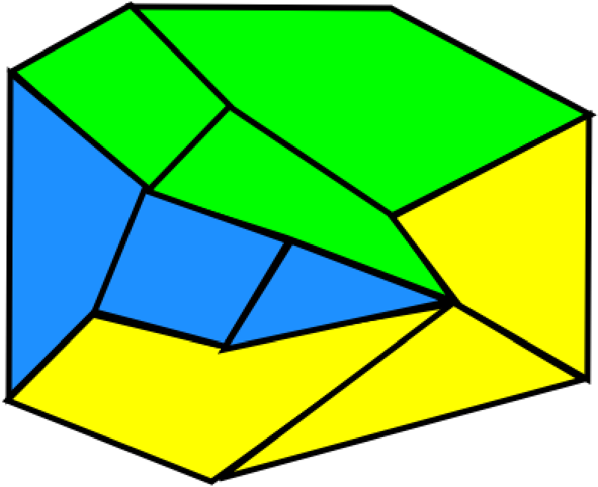

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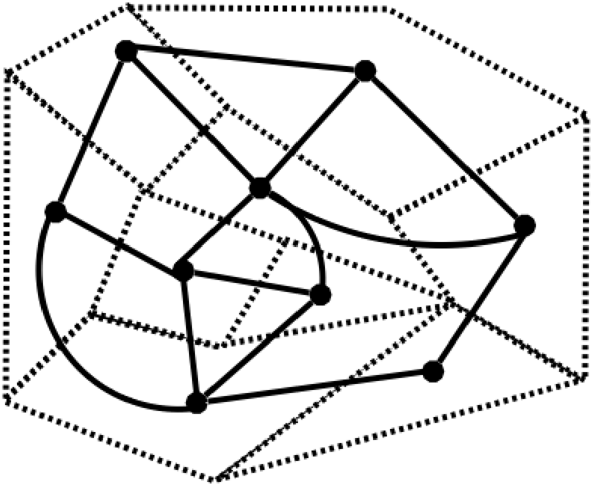

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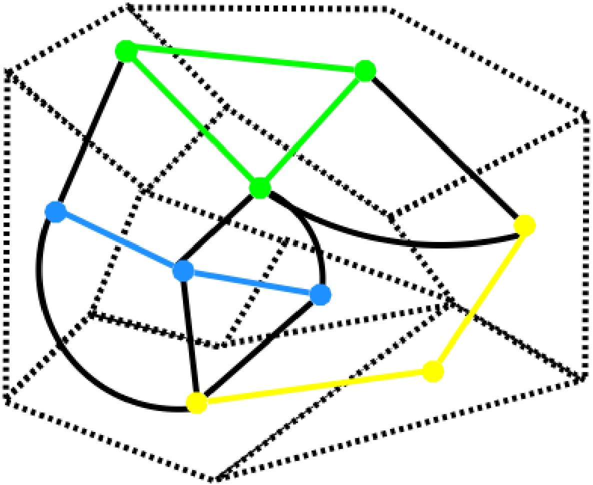

Györi and Lovász Theorem

Theorem (Györi and Lovász)

A graph G with n vertices is k-connected iff for any subset $\{v_1, v_2, \ldots, v_k\}$ of k vertices, and any positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$, there exists a partition of $V(G)$ into k parts $V_1, V_2, \ldots, V_k$ such that $v_i \in V_i$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for all $1 \leq i \leq k$.

k-Partitionable and Decomposable Graphs

Definition

A graph G with n vertices is said to be k-partitionable if for all positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$, there exists a partition of $V(G)$ into k parts $V_1, V_2, \ldots, V_k$ such that $|V_i| = n_i$ and V_i induces a connected subgraph of G, for $1 \leq i \leq k$.

Definition

A graph G is said to be decomposable if it is k-partitionable for all $k \geq 1$.

Algorithmic Complexity

- NP-Hard to find a k-partition of an arbirary graph, for all $k \geq 2$.
- No polynomial-time algorithm known to find a k-partition for a k-connected graph for $k \geq 4$. The partition always exists by the Györi-Lovász Theorem.
- NP-Hard to recognize k-partitionable and decomposable graphs, for $k \geq 2$.
- Not clear whether recognizing k-partitionable and decomposable graphs is in NP, for arbitrary k.

Sufficient Conditions for k-Partitionability

- k-connected graphs are k-partitionable for all $k \geq 1$. (Györi-Lovász Theorem).
- k-connected graphs are not $(k+1)$-partitionable in general.
- Complete bipartite graph $K_{k,k+2}$ has no perfect matching.
- Does k-connectivity with some additional property imply higher partitionability?

Planar Graphs

- $K_{1,3}$ is a planar 1-connected graph that is not 2-partitionable.
- $K_{2,4}$ is a planar 2-connected graph that is not 3-partitionable.
- Planar 4-connected graphs are Hamiltonian (Tutte's Theorem), which implies they are decomposable.
- What happens for 3-connected planar graphs? ($K_{3,5}$ is not planar).
- Conjecture: Planar 3-connected graphs are 6-partitionable.

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Plane Triangulations

Definition

A plane triangulation is a planar simple graph in which every face is a triangle. Equivalently, it is a maximal planar graph with at least 3 vertices.

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Plane triangulations are 6-partitionable.

The proof also gives a polynomial-time algorithm to find a 6-partition.

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Plane Near-Triangulations

Definition

A plane near-triangulation is a planar simple graph in which all internal (bounded) faces are triangles and the outer face is a simple cycle.

Theorem

Plane near-triangulations are 4-partitionable.

Contractible Edges in Triangulations

Lemma

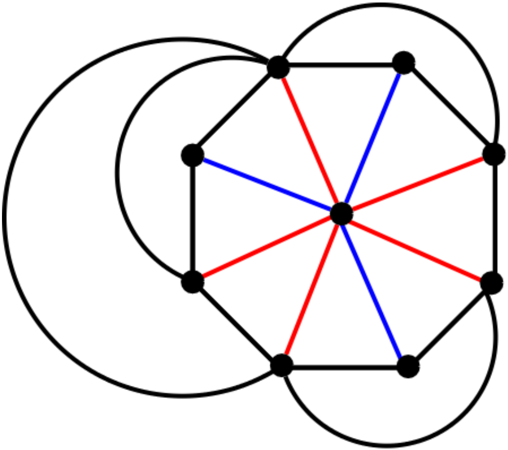
Let u, v, w be the vertices on the boundary of some face of a plane triangulation with at least 4 vertices. There exists a vertex $x \notin \{v, w\}$ such that contracting edge ux gives a plane triangulation.

Contractible Edges in Triangulations

Lemma

Let u be any vertex in a plane triangulation with at least 4 vertices. There are at least two edges uv, uw incident with u such that contracting uv or uw gives a plane triangulation.

Contractible Edges



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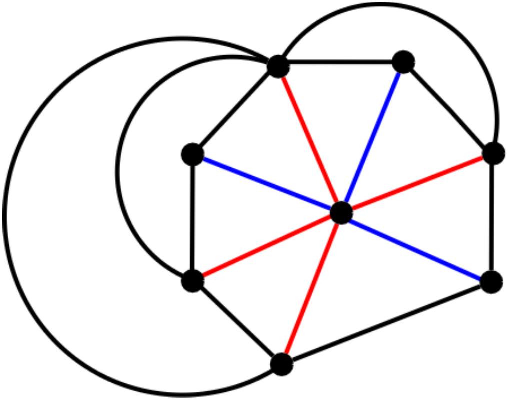

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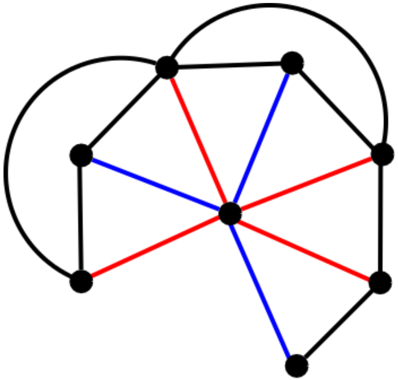


Contractible Edges in Chordless Near-Triangulations

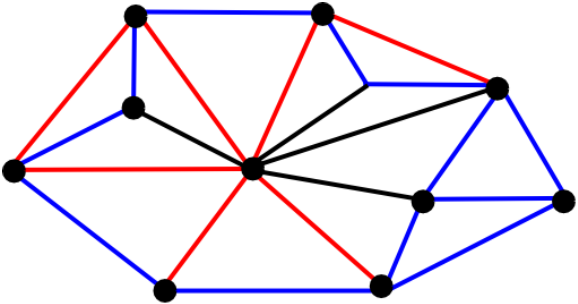
Lemma

Let u be a vertex in the external cycle of a chordless near-triangulation G with at least 4 vertices. Then at least one of the following holds:

(i) *There exists an internal vertex x adjacent to u such that contracting the edge ux gives a chordless near-triangulation.*

(ii) *Contracting any external edge incident with u gives a chordless near-triangulation.*

Contractible Edges

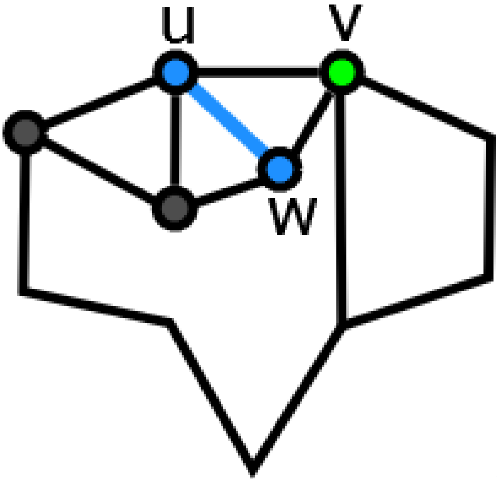


3-Partitioning Near-Triangulations

Lemma

Let G be a plane near-triangulation with n vertices and let u, v be two adjacent vertices in the outer face of G. Then for any 3 positive integers n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = n$, there exists a partition of $V(G)$ into 3 parts V_1, V_2, V_3 such that $u \in V_1$, $v \in V_2$, $|V_i| = n_i$ and $G[V_i]$ is connected, for $1 \leq i \leq 3$.

3-Partitioning Near-Triangulations



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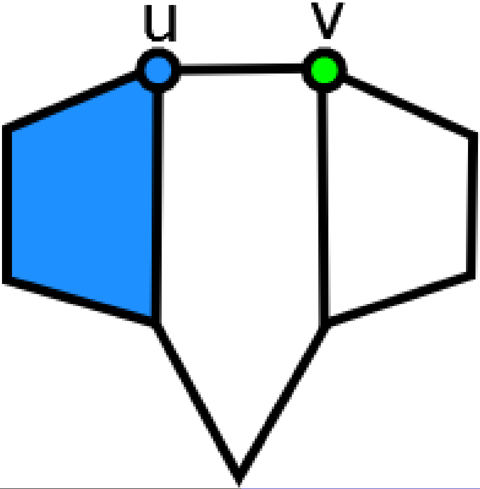

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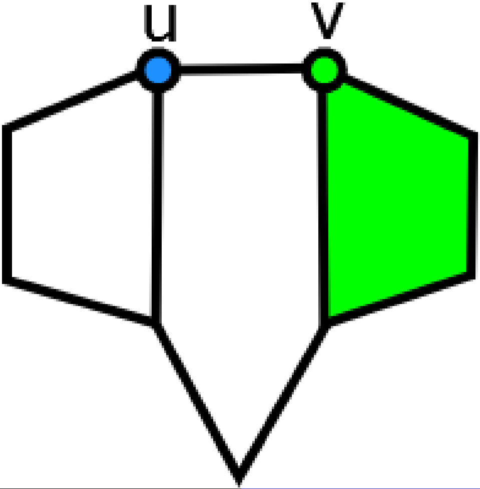

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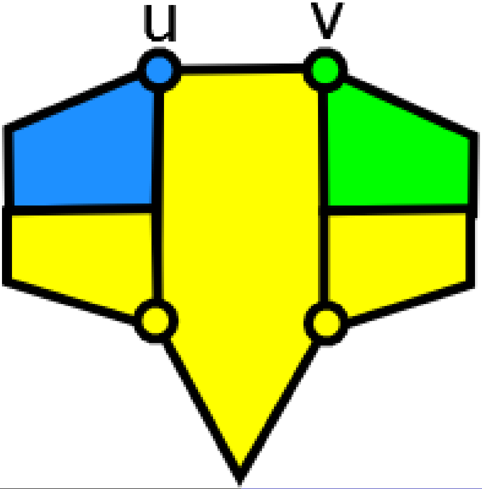

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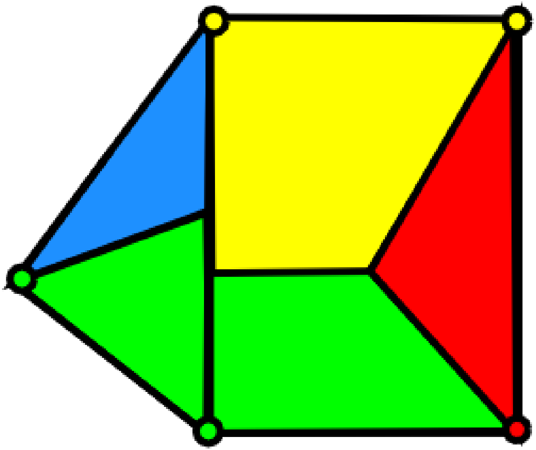

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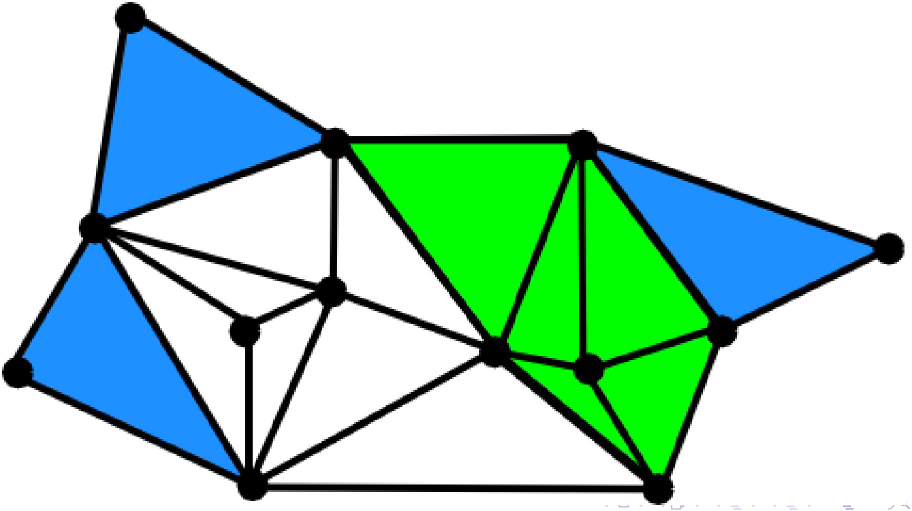
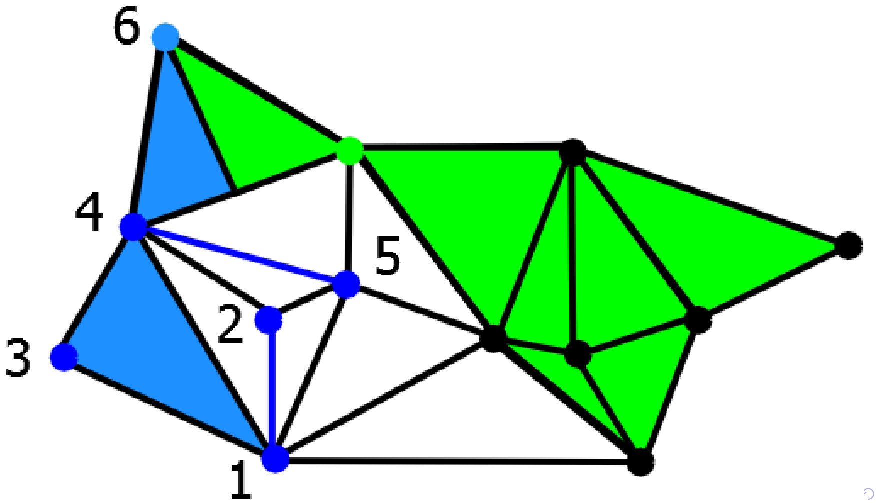


4-Partitioning Near-Triangulations



4-Partitioning a Plane Triangulations

Lemma

Let u, v, w be vertices on the boundary of some face of a plane triangulation G with n vertices. Then for all positive integers n_1, n_2, n_3, n_4 such that $n_1 + n_2 + n_3 + n_4 = n$, there exists a partition of $V(G)$ into parts V_1, V_2, V_3, V_4, such that $u \in V_1$, $v \in V_2$, $w \in V_3$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for $1 \leq i \leq 4$.

4-Partitioning a Plane Triangulation

Lemma

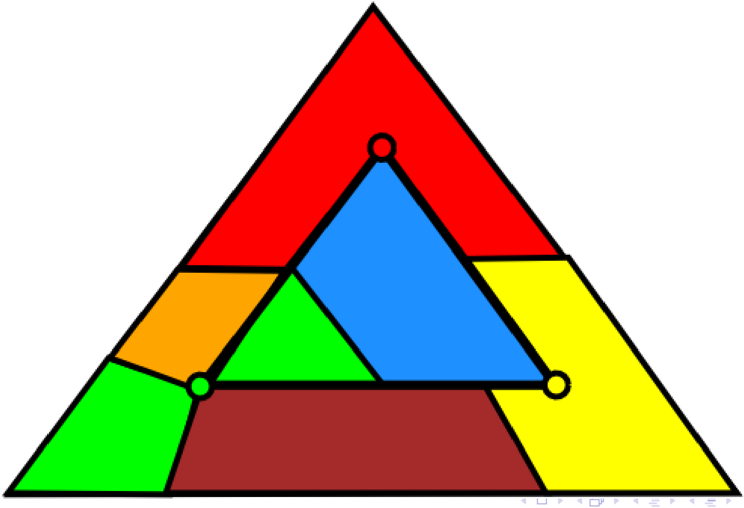
Let G be a plane triangulation with n vertices and let u, v, w be the vertices on the boundary of some face in G. Then for all positive integers n_1, n_2, n_3, n_4 such that $n_1 + n_2 + n_3 + n_4 = n - 1$, there exists a partition of $V(G) - v$ or $V(G) - w$ into parts V_1, V_2, V_3, V_4, such that $u \in V_1$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for $1 \leq i \leq 5$.

5-Partitioning a Plane Triangulation

Lemma

Let u be any vertex in a plane triangulation G with n vertices. Then for all positive integers n_1, n_2, n_3, n_4, n_5 such that $n_1 + n_2 + n_3 + n_4 + n_5 = n$, there exists a partition of $V(G)$ into parts V_1, V_2, V_3, V_4, V_5 such that $u \in V_1$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for $1 \leq i \leq 5$.

6-Partitioning a Plane Triangulation



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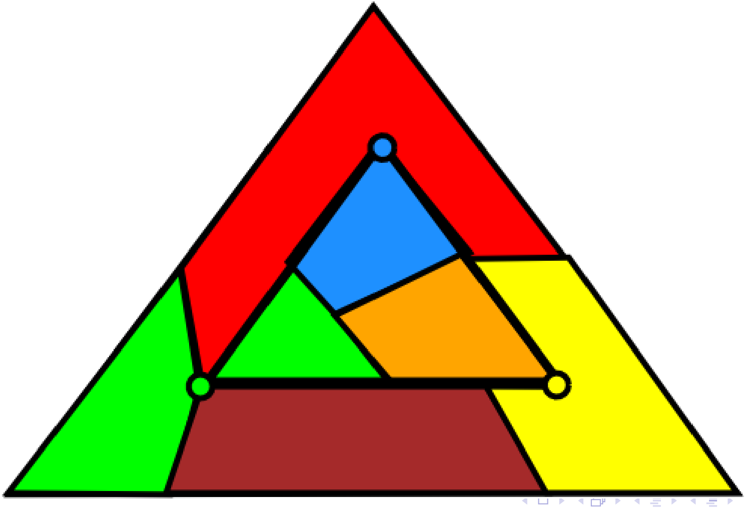

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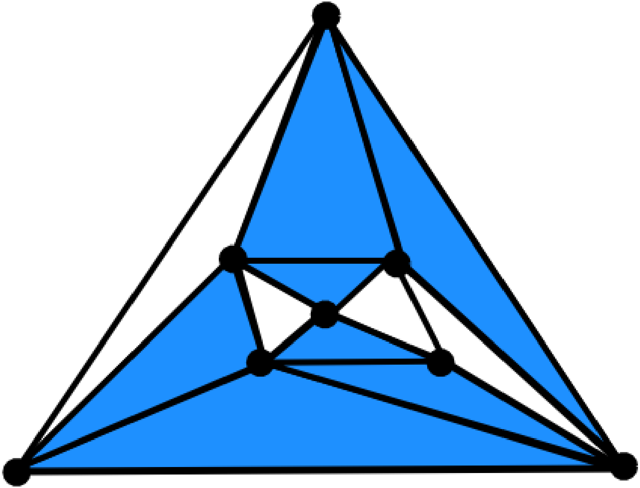

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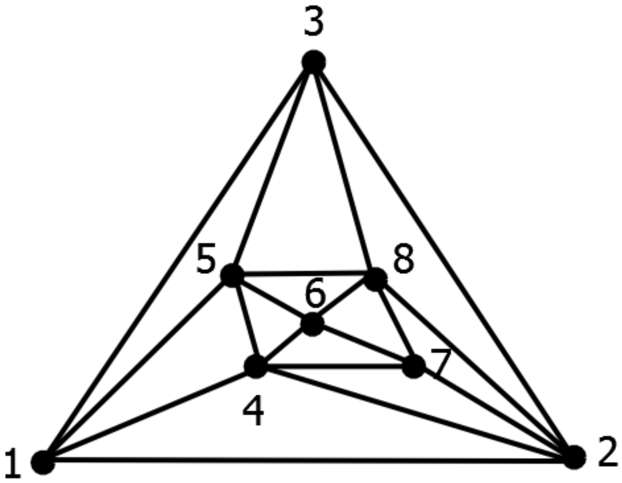

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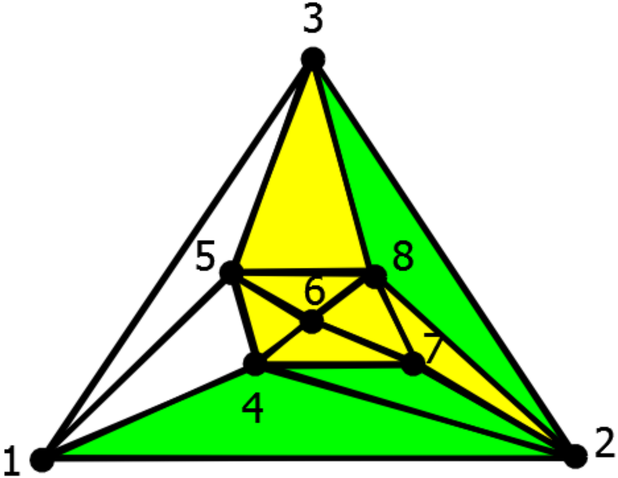

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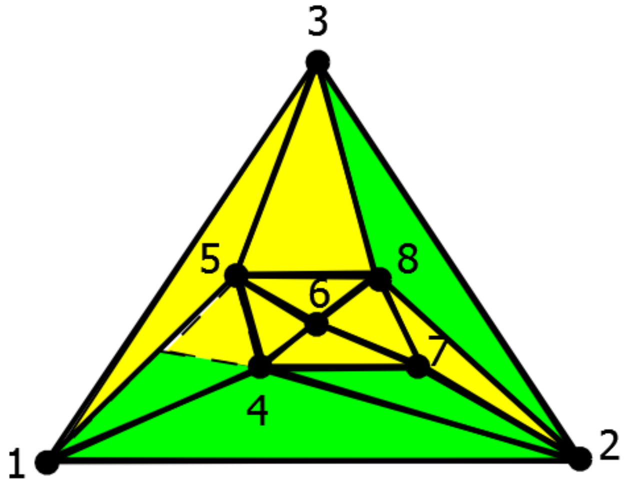
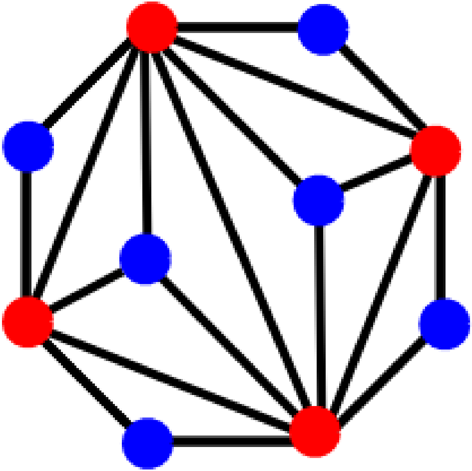
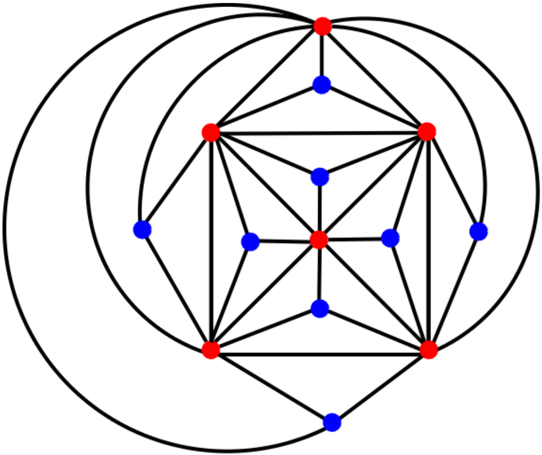


Near-Triangulations are not 5-partitionable



Triangulations are not 7-partitionable



Conjecture

Planar 3-connected graphs are 6-partitionable.

Partitioning 2-connected Graphs

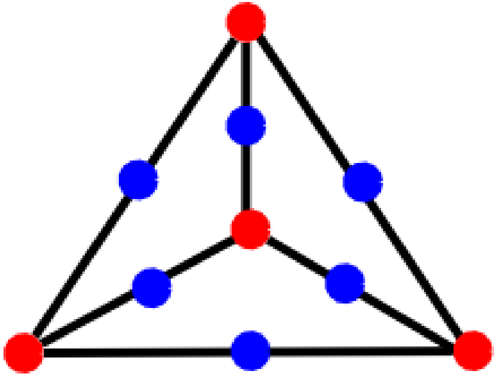
Theorem

Every 2-connected graph with maximum degree at most 3 is 4-partitionable.

Theorem

Every 2-connected claw-free ($K_{1,3}$-free) graph is 4-partitionable.

Counterexamples



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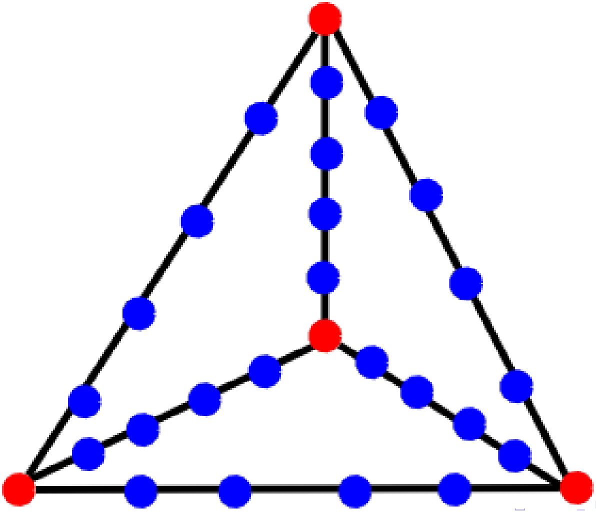

Partitioning k-connected Graphs

Is every k-connected graph with maximum degree at most $k+1$ $2k$-partitionable?

Is every k-connected k-regular graph decomposable, that is, l-partitionable for all $l \geq 1$.

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Thank You