Art Gallery Theorems and Approximation Algorithms

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Overview

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3. Art gallery theorems on stationary guards
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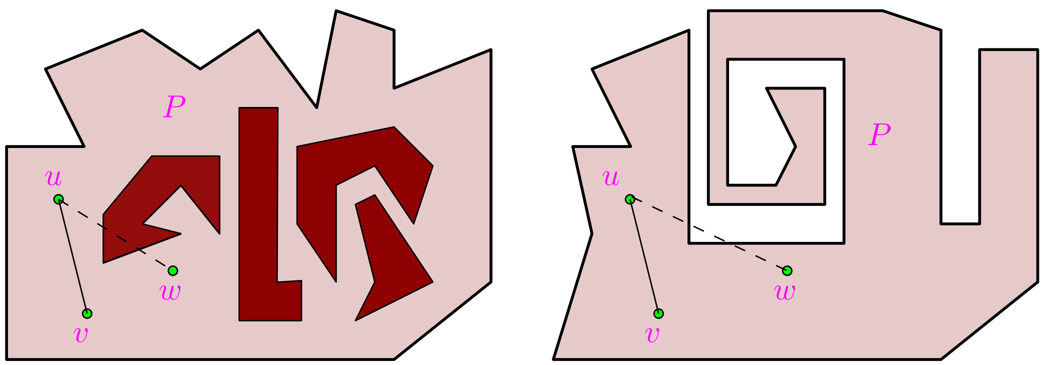
Polygon



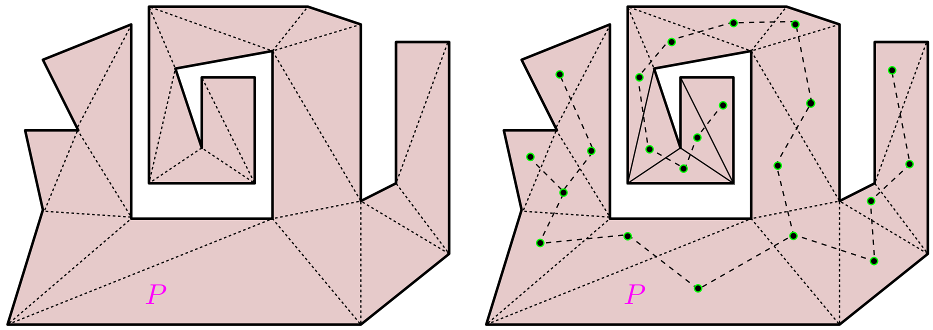
A *polygon* P is defined as a closed region in the plane bounded by a finite set of line segments (called *edges* of P) such that there exists a path between any two points of P which does not intersect any edge of P.

If the boundary of P consists of two or more cycles, then P is called a *polygon with holes*. Otherwise, P is called a *simple polygon* or a *polygon without holes*.

Two points u and v in a polygon P are said to be *visible* if the line segment joining u and v lies inside P.

Polygon triangulation

A line segment joining any two mutually visible vertices of a polygon is called a *diagonal* of the polygon.



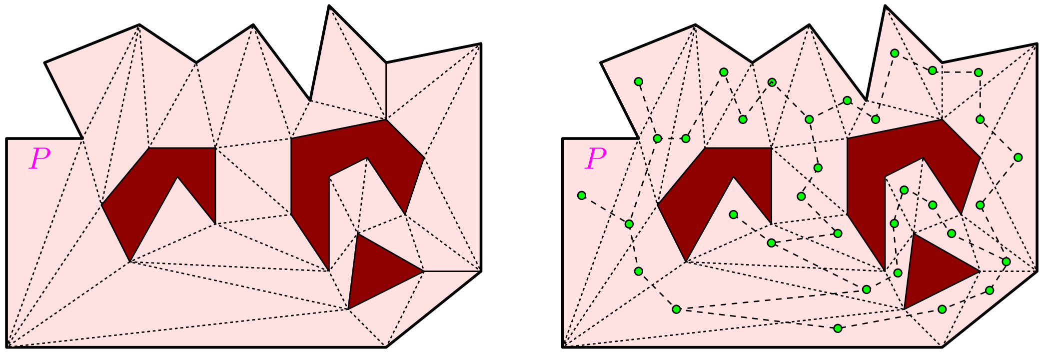
Lemma: Every triangulation of a simple polygon P of n vertices uses $n-3$ diagonals and has $n-2$ triangles.

Corollary: The sum of the internal angles of a simple polygon of n vertices is $(n-2)\pi$.

Lemma: The dual of a triangulation of P is a tree.

Lemma: Every polygon P has at least two disjoint ears.

1. B. Chazelle, *Triangulating a simple polygon in linear time*, Discrete and Computational Geometry, 6 (1991), 485–524.
2. G. H. Meisters, *Polygons have ears*, American Mathematical Monthly, 82 (1975), 648-651.



Lemma: Every triangulation of a polygon with h holes with a total of n vertices uses $n+3h-3$ diagonals and has $n+2h-2$ triangles.

Lemma: The dual graph of a triangulation of a polygon with holes must have a cycle.

1. J. O'Rourke, *Computational Geometry in C*, Cambridge University Press, 1994 (Second Edition in 1998).

2. S. K. Ghosh and D. M. Mount, *An output sensitive algorithm for computing visibility graphs*, SIAM Journal on Computing, 20 (1991) 888-910. Running time for triangulation is $O(n \log n)$.

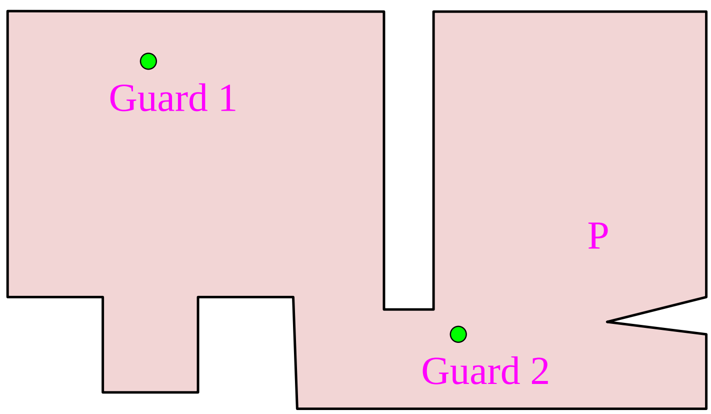
3. R. Bar-Yehuda, B. Chazelle, *Triangulating disjoint Jordan chains*, International Journal of Computational Geometry and Applications, 4 (1994) 475-481. Running time for triangulation is $O(n + h \log^{1+\epsilon} h)$, for $\epsilon > 0$.

The art gallery problem

An art gallery can be viewed as a polygon P with or without holes with a total of n vertices and guards as points in P.

During a conference at Stanford in 1976, Victor Klee asked the following question:

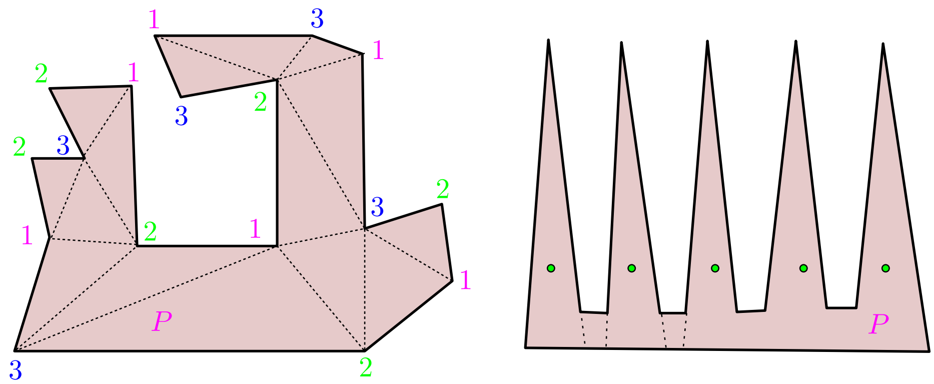
How many guards are always sufficient to guard any polygon with n vertices?



1. R. Honsberger, *Mathematical games II*, Mathematical Associations for America, 1979.

Chvatal's art gallery theorem

Theorem: A simple polygon P of n vertices needs at most $\lfloor \frac{n}{3} \rfloor$ guards.

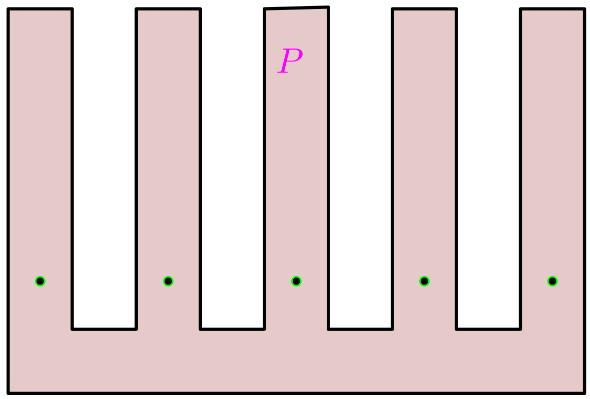


Lemma: All vertices of P can be coloured using three colours (say, $\{1, 2, 3\}$) such that two vertices joined by an edge of P or by a diagonal in the triangulation of P receive different colours.

1. V. Chvatal, *A combinatorial theorem in plane geometry*, Journal of Combinatorial Theory, Series B, 18 (1975), 39-41.
2. S. Fisk, *A short proof of Chvatal's watchman theorem*, Journal of Combinatorial Theory, Series B, 24 (1978), 374.

KKK's art gallery theorem

Theorem: An orthogonal polygon P of n vertices needs at most $\lfloor \frac{n}{4} \rfloor$ guards.

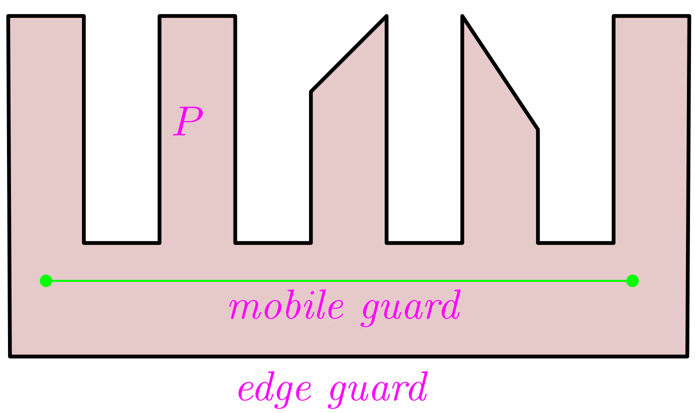


1. J. Kahn, M. Klawe and D. Kleitman, *Traditional galleries require fewer watchmen*, SIAM Journal of Algebraic and Discrete Methods, 4 (1983), 194-206.

2. J. O'Rourke, *An alternative proof of the rectilinear art gallery theorem*, Journal of Geometry, 211 (1983), 118-130.

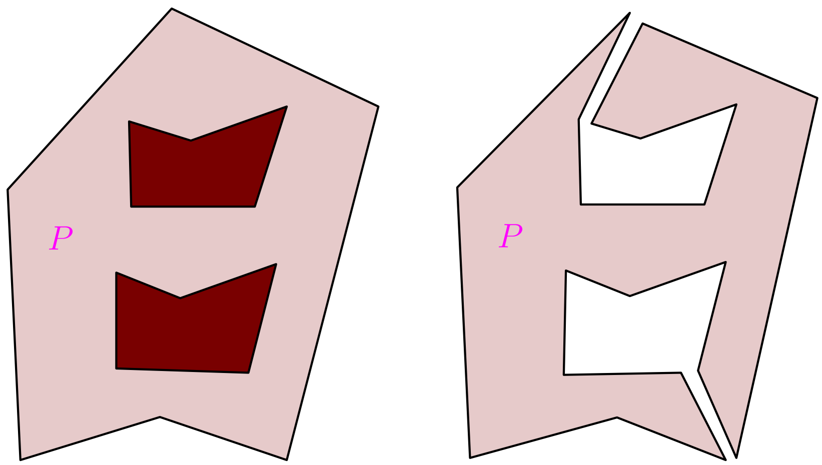
Different types of guards

- *Point guards*: These are guards that are placed anywhere in the polygon.
- *Vertex guards*: These are guards that are placed on vertices of the polygon.
- *Edge guards*: These are guards that are allowed to patrol along an edge of the polygon.
- *Mobile guards*: These are guards that are allowed to patrol along a segment lying inside a polygon.



Art Gallery theorems in polygons with holes

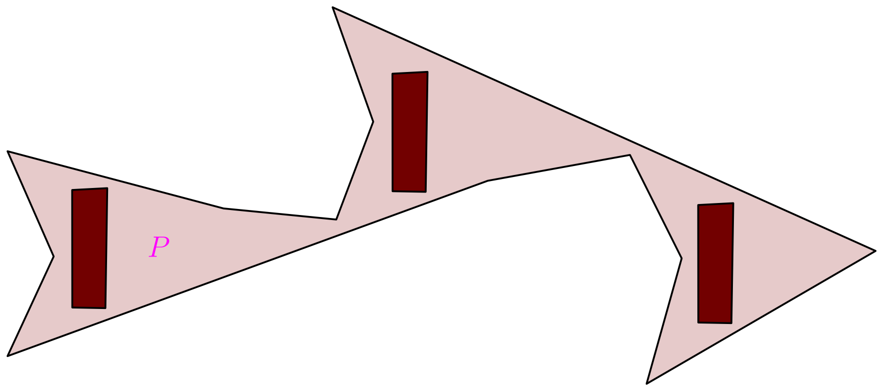
Theorem: Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards.



Conjecture: (Shermer) Any polygon P with n vertices and h holes can always be guarded with $\lfloor \frac{n+h}{3} \rfloor$ vertex guards.

The conjecture has been proved by Shermer for $h = 1$. For $h > 1$, the conjecture is still open.

1. J. O'Rourke, *Art gallery theorems and algorithms*, Oxford University Press, 1987.



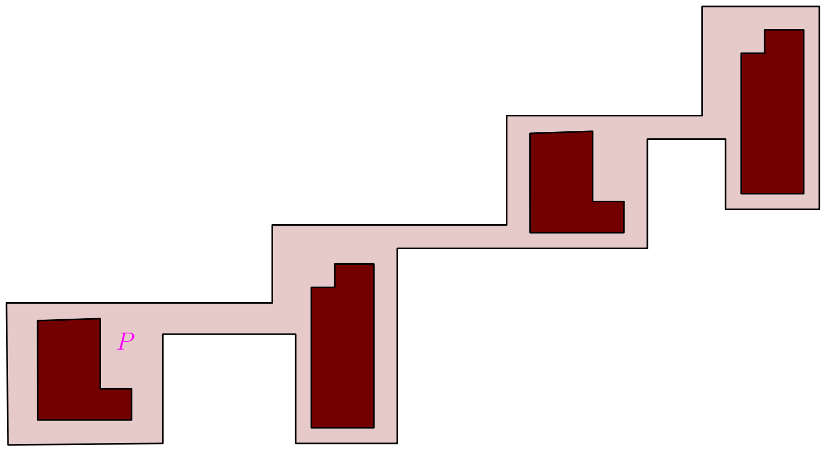
A polygon of 24 vertices with 3 holes that requires 9 guards.

Theorem: To guard a polygon P with n vertices and h holes, $\lceil \frac{n+h}{3} \rceil$ point guards are always sufficient and occasionally necessary.

1. I. Bjorling-Sachs and D. L. Souvaine, *An efficient algorithm for guard placement in polygons with holes*, Discrete and Computational Geometry 13 (1995), 77-109.

2. F. Hoffmann, M. Kaufmann and K. Kriegel, *The art gallery theorem for polygons with holes*, Proceedings of the 32nd Symposium on the Foundation of Computer Science, 39-48, 1991.

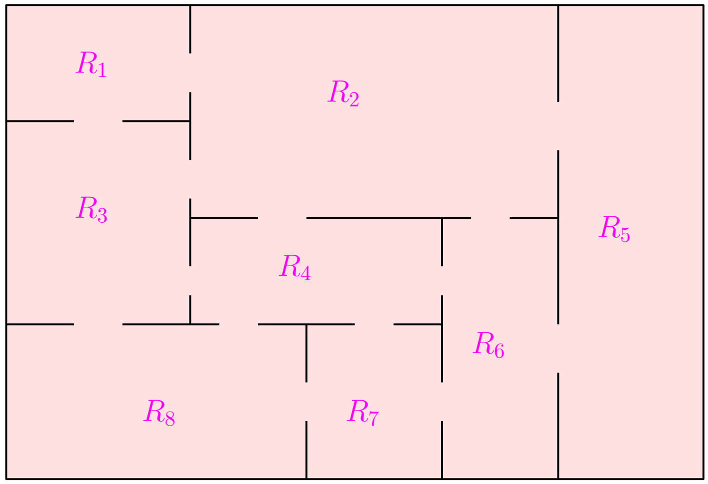
Theorem: To guard an orthogonal polygon P with n vertices and h holes, $\lfloor \frac{n}{4} \rfloor$ point guards are always sufficient.



A polygon of 44 vertices with 4 holes that requires 12 guards.

Theorem: To guard an orthogonal polygon P with n vertices and h holes, $\lfloor \frac{n}{3} \rfloor$ vertex guards are always sufficient.

1. F Hoffmann, *On the rectilinear art gallery problem*, Proceedings of ICALP, Lecture Notes in Computer Science, 90 (1990), 717-728, Springer-Verlag.
2. F. Hoffmann and K. Kriegel, *A graph-coloring result and its consequences for polygon-guarding problems*, SIAM Journal on Discrete Mathematics, 9 (1996), 210-224.

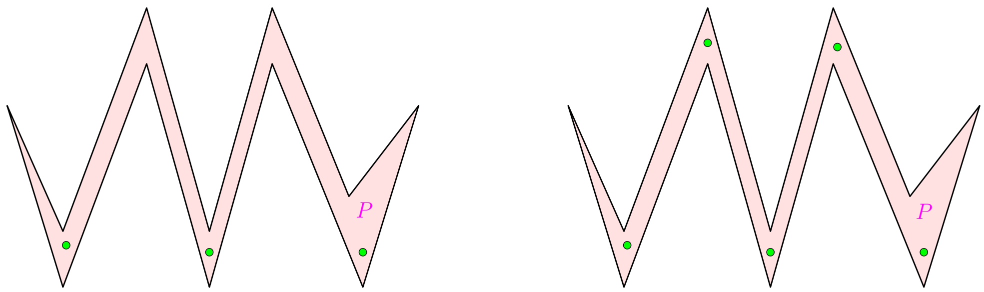


Theorem: Any rectangular art gallery with n rooms can be guarded with exactly $\lceil \frac{n}{2} \rceil$ guards.

1. J. Czyzowicz, E. Rivera-Campo, N. Santoro, J. Urrutia and J. Zaks, *Guarding rectangular art galleries*, Discrete Applied Mathematics, 50 (1994), 149-157.

Art Gallery theorems on cooperative guards

In the standard art gallery problem, $\lfloor \frac{n}{3} \rfloor$ stationary guards are sufficient and sometime necessary for guarding P containing no holes.



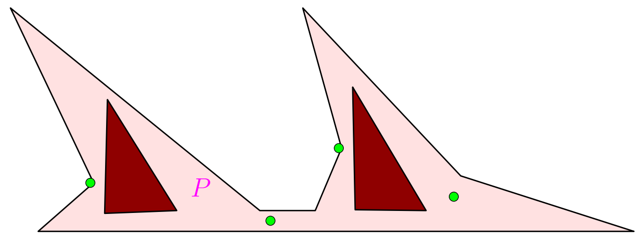
Suppose, guards g_1, g_2,...., g_k are placed in P for security reasons in a such way that each guard g_i for $i > 1$ is visible at least from one other guard g_j for $i < j$. In that case, $\lfloor \frac{n}{3} \rfloor$ guards are not sufficient.

Theorem: To guard a simple polygon P with n vertices, $\lfloor \frac{n}{2} \rfloor - 1$ cooperative guards are necessary and sufficient.

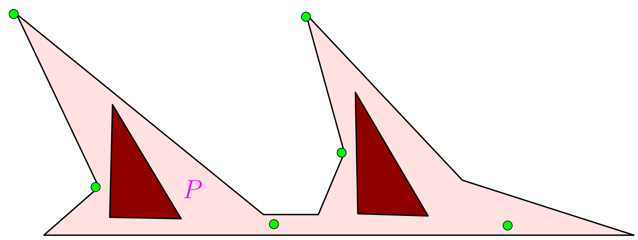
1. G. Hernandez-Penalver, *Controlling guards*, Proceedings of the Sixth Canadian Conference on Computational Geometry, pp. 387-392, 1994.

2. T.S. Michael and V. Pinciu, *Art gallery theorems for guarded guards*, Computational Geometry: Theory and Applications, 26 (2003), 247-258.

In the standard art gallery problem for a polygon P contains h holes, $\lfloor \frac{n+h}{3} \rfloor$ stationary guards are sufficient and sometime necessary.



If the guards also have to satisfy the visibility constraint of cooperative guards as stated above, then $\lfloor \frac{n+h}{3} \rfloor$ guards are not sufficient.



Conjecture: (Ghosh et al.) Any polygon P with n vertices and h holes can be guarded with $\lfloor \frac{n+2h}{3} \rfloor$ cooperative guards.

1. S. K. Ghosh, J. W. Burdick, A. Bhattacharya and S. Sarkar, *On-line algorithms with discrete visibility: Exploring unknown polygonal environments*, IEEE Robotics and Automation Magazine, 15:2 (2008) 67-76.

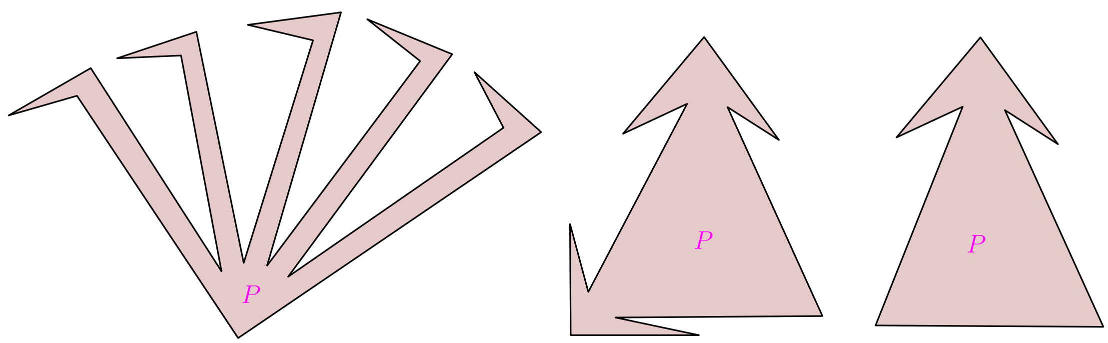
2. J. Urrutia, Art Gallery and illumination problems, Handbook of Computational Geometry (Ed. J.-R. Sack and J. Urrutia), Elsevier Science, pp. 973-1027, 2000.

Art Gallery theorems on mobile guards

In 1981, Godfried Toussaint asked the following question:

How many edge/mobile guards are always sufficient to guard any polygon with n vertices?

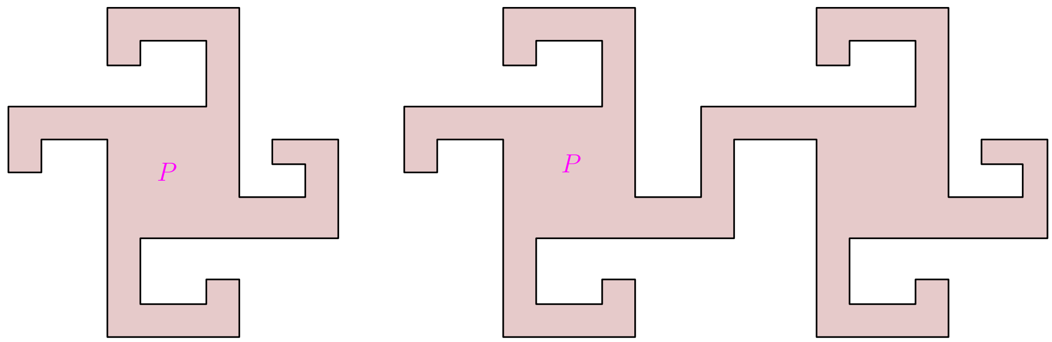
Theorem: To guard a simple polygon with n vertices, $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient and occasionally necessary.



Conjecture: (Toussaint) Except for a few polygons, $\lfloor \frac{n}{4} \rfloor$ edge guards are always sufficient to guard any simple polygon with n vertices.

1. J. O'Rourke, *Galleries need fewer mobile guards: A variation on Chvatal's theorem*, Geometricae Dedicata, 4 (1983), 273-283.

Theorem: To guard an orthogonal polygon with n vertices, $\lfloor \frac{3n+4}{16} \rfloor$ mobile or edge guards are always sufficient and occasionally necessary.



Theorem: To guard an orthogonal polygon with n vertices and h holes, $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are always sufficient and occasionally necessary.

1. A. Aggarwal, *The art gallery theorems: Its variations, applications and algorithmic aspects*, Ph. D. thesis, John Hopkins University, 1984.
2. I. Bjorling-Sachs, *Edge guards in rectilinear polygons*, Computational Geometry: Theory and Applications, 11 (1998), 111-123.
3. E. Gyri, F. Hoffmann, K. Kriegel and T. Shermer, *Generalized guarding and partitioning for rectilinear polygons*, Computational Geometry: Theory and Applications, 6 (1996), 21-44.

NP-complete problems

A problem belongs to a class P if the problem can be solved in $O(n^m)$ time (i.e. in deterministic polynomial time), where n is the problem size and m is some constant.

A problem belongs to a class NP if the problem can be solved in non-deterministic polynomial time.

Polynomial reducibility: If a problem A can be transformed in deterministic polynomial time to an instance of a problem B such that the solution to B yields a solution to A, then A is said to be polynomial reducible to B.

If A is polynomial reducible to B, then B is as hard as A, for B can be solved in polynomial time, so can A.

Though $P \subseteq NP$ follows from the definitions, no one has been able to prove $P \neq NP$, which is a major open problem.

1. M.R. Garey and D.S. Johnson, *Computer and Intractability: A guide to the theory of NP-completeness*, W.H. Freeman and Company, 1979.
2. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, MIT Press, Cambridge, 1991.

Minimum number of guards

Let P be a polygon with or without holes. What is the minimum number of guards required for guarding P?

Suppose, a positive integer k is given. Can it be decided in polynomial time whether k guards are sufficient to guard P?

The problem is NP-complete.

Theorem: The minimum vertex, point and edge guard problems for polygons with or without holes (including orthogonal polygons) are NP-hard.

1. D. T. Lee and A. K. Lin, *Computational Complexity of Art Gallery Problems*, IEEE Transactions on Information Theory, IT–32 (1986), 276–282.
2. J. O'Rourke and K. Supowit, *Some NP-hard polygon decomposition problems*, IEEE Transactions on Information Theory, IT-29 (1983), 181-190.
3. D. Schuchardt and H. Hecker, *Two NP-hard art-gallery problems for ortho-polygons*, Mathematical Logic Quarterly, 41 (1995), 261-267.
4. B.C. Liaw, N.F. Huang, R.C.T. Lee, *The minimum cooperative guards problem on k-spiral polygons*, Proceedings of the Canadian Conference on Computational Geometry, pp. 97–101, 1993.

Approximation algorithms for minimum guard problems

An algorithm that returns sub-optimal solutions for an optimization problem is called an *approximation algorithm*.

Approximation algorithms are often associated with NP-hard problems; since it is unlikely that there can ever be efficient polynomial time exact algorithms solving NP-hard problems, one settles for polynomial time sub-optimal solutions.

We present polynomial time approximation algorithms for the minimum vertex and edge guard problems in a polygon P with or without holes.

1. S. K. Ghosh, *Approximation algorithms for art gallery problems*, Technical report no. JHU/EECS-86/15, Department of Electrical Engineering and Computer Science, The Johns Hopkins University, August 1986. Also in Proceedings of the Canadian Information Processing Society Congress, pp. 429-434, 1987. Running time: $O(n^5 \log n)$ time. Approximation ratio: $O(\log n)$.

2. S. K. Ghosh, *Approximation algorithms for art gallery problems in polygons*, Manuscript, Tata Institute of Fundamental Research, 2007. Recently, the running time has been improved to $O(n^4)$ for simple polygons and $O(n^5)$ for polygons with holes, keeping the approximation ratio same.

3. A. Aggarwal, S. K. Ghosh and R. K. Shyamasundar, Computational complexity of restricted polygon decompositions, Computational Morphology (edited by G.T.Toussaint), Machine Intelligence and Pattern Recognition, vol. 6, pp. 1-11, 1988. Running time: $O(n^4 \log n)$ time. Approximation ratio: $O(\log n)$.

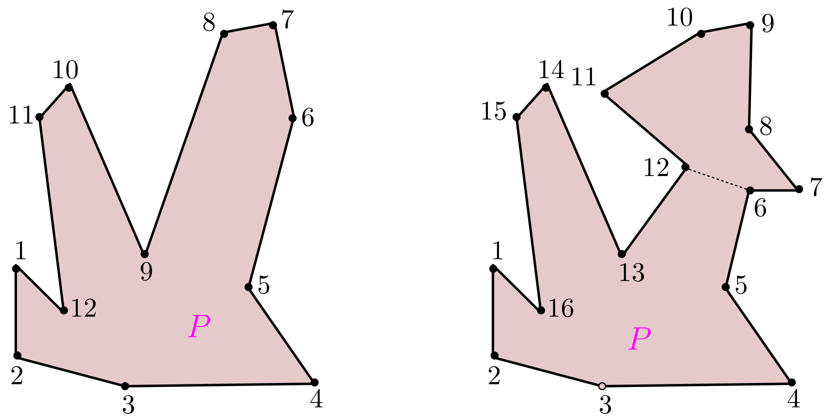
4. A. Efrat and S. Har-Peled, *Guarding galleries and terrains*, Information Processing Letters, 100 (2006), 238-245. (i) For simple polygons, $O(nc_{opt}^2 \log^4 n)$ expected time, and $O(\log c_{opt})$ approximation ratio, where c_{opt} is the number of vertices in the optimal solution. (ii) For polygons with h holes, $O(nhc_{opt}^3 polylog\, n)$ expected time, and $O(\log n \log(c_{opt} \log n))$ approximation ratio.

5. B. Ben-Moshe, M. Katz and J. Mitchell, *A constant-factor approximation algorithm for optimal terrain guarding*, SIAM Journal on Computing, 36 (2007), 1631-1647. Running time: $O(n^4)$ time. Approximation ratio: $O(1)$.

6. J. King, *A 4-approximation algorithm for guarding 1.5-dimensional terrain*, LATIN 2006, LNCS, Springer-Verlag, no. 3887, pp. 629-640, 2007. Running time: $O(n^2)$ time. Approximation ratio: 5.

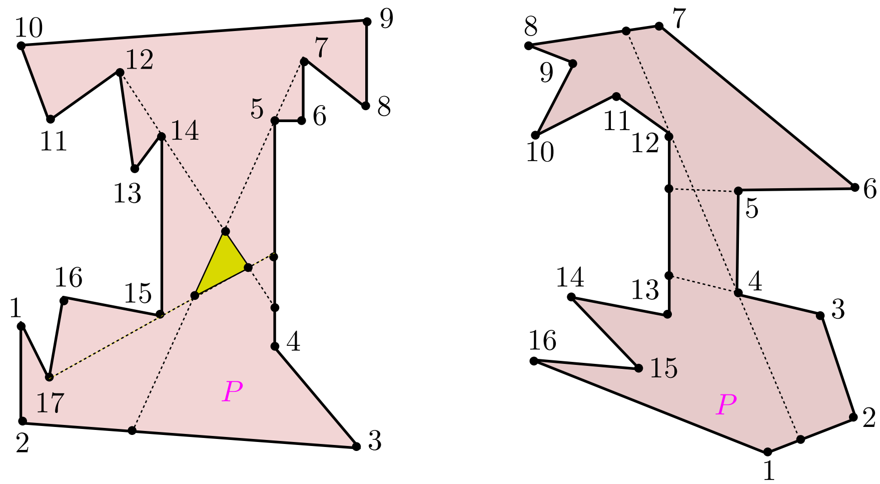
7. A. Deshpande, T. Kim, E. D. Demaine1 and S. E. Sarma, *A pseudopolynomial time O(log n)-approximation algorithm for art gallery problems* , Proceedings of the 10th International Workshop on Algorithms and Data Structures, LNCS, Springer-Verlag, no. 4619, pp. 163-174, 2007. Running time: Polynomial in n, the number of walls and the spread, where the spread can be exponential. Approximation ratio: $O(\log c_{opt})$.

Vertex-guard problem



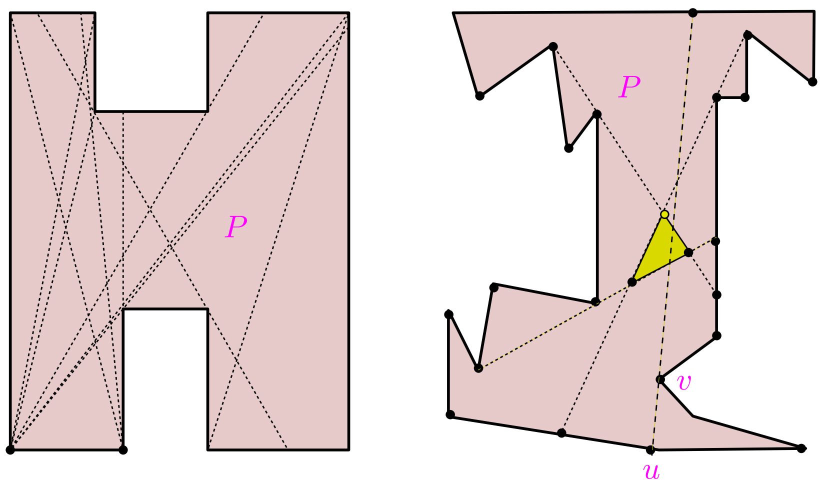
A simple polygon is called a *fan* if there exists a vertex that is visible from all points in the interior of the polygon.

The vertex guard problem can be treated as a polygon decomposition problem in which the decomposition pieces are fans.



Vertices 7, 12 and 17 together can see the entire boundary of the polygon but the shaded region is not visible from any of these vertices.

Three fans (vertices 1, 4 and 7) are necessary to cover the polygon if only edge extensions are allowed, whereas two fans (vertices 1 and 7) suffice if we allow the boundary of convex components to be bounded by segments that contains any two vertices of the polygon.

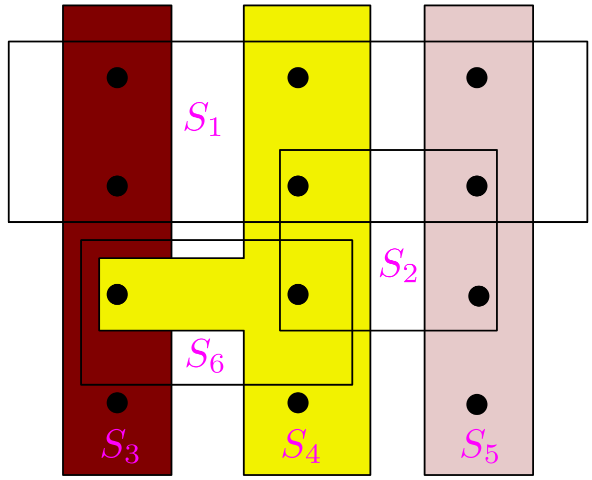


The polygonal region of P is decomposed into convex components where every component is bounded by segments that contains any two vertices of the polygon.

Every convex component must lie in at least one of the fans chosen by the approximation algorithm.

Lemma: Every convex component is either totally visible or totally invisible from a vertex of P.

Minimum set-covering problem



Given a finite family C of sets $S_1, \ldots, S_n$, the problem is to determine the minimum cardinality $A \subseteq C$ such that $\bigcup_{i \in A} S_i = \bigcup_{j=1}^{n} S_j$.

The problem of finding the minimum number of fans to cover P is same as the minimum set-covering problem, where every fan is a set and convex components are elements of the set.

1. D. S. Johnson, *Approximation algorithms for combinatorial problems*, Journal of Computer and System Sciences, 9 (1974), 256-278.

Vertex-guard algorithm

Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.

Step 2: *For* $1 \leq j \leq n$, construct the set F_j by adding those convex components of P that are totally visible from the vertex v_j.

Step 3: Find $i \in N$ such that $|F_i| \geq |F_j|$ *for all* $j \in N$ and $i \neq j$.

Step 4: Add i to Q and delete i from N.

Step 5: *For all* $j \in N$, $F_j := F_j - F_i$, and $C := C - F_i$.

Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.

Step 7: Output the set Q and Stop.

Theorem: The approximation algorithm for the minimum vertex guard problem in a polygon P of n vertices computes solutions that are at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

Edge-guard algorithm

Step 1: Draw lines through every pair of vertices of P and compute all convex components $c_1, c_2, \ldots, c_m$ of P. Let $C = (c_1, c_2, \ldots, c_m)$, $N = (1, 2, \ldots, n)$ and $Q = \emptyset$.

Step 2: *For* $1 \leq j \leq n$, construct the set E_j by adding those convex components of P that are totally visible from the edge e_j of P.

Step 3: Find $i \in N$ such that $|E_i| \geq |E_j|$ *for all* $j \in N$ and $i \neq j$.

Step 4: Add i to Q and delete i from N.

Step 5: *For all* $j \in N$, $E_j := E_j - E_i$, and $C := C - E_i$.

Step 6: *If* $|C| \neq \emptyset$ *then goto* Step 3.

Step 7: Output the set Q and Stop.

Theorem: For the minimum edge guard problem in an n-sided polygon P, an approximate solution can be computed which is at most $O(\log n)$ times the optimal. If P is a simple polygon, the approximation algorithm runs in $O(n^4)$ time. If P is a polygon with holes, the approximation algorithm runs in $O(n^5)$ time.

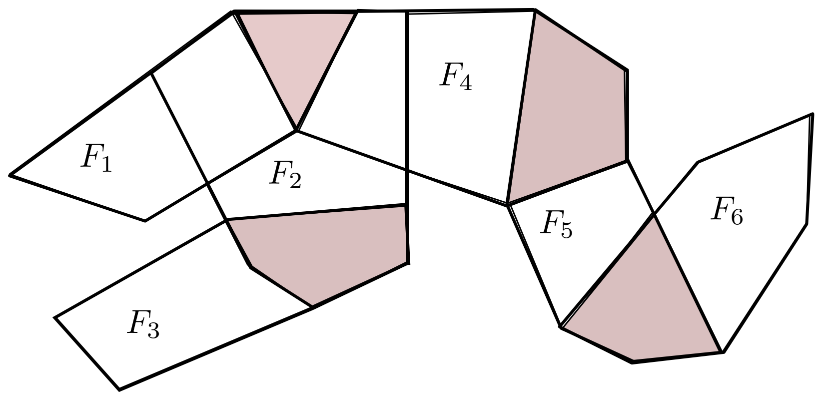
Upper bound on approximation ratio

a_1	a_2	a_3	a_4	a_5	a_6
b_1	b_2	b_3			
			c_4	c_5	c_6
d_1	d_2				
		e_3	e_4		
				f_5	f_6
g_1					
	h_2				
		i_3			
			j_4		
				k_5	
					l_6

Sets corresponding to rows are $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, $\{b_1, b_2, b_3\}$, $\{c_4, c_5, c_6\}$, $\{d_1, d_2\}$, $\{e_3, e_4\}$, $\{f_5, f_6\}$, $\{g_1\}$, $\{h_2\}$, $\{i_3\}$, $\{j_4\}$, $\{k_5\}$, $\{l_6\}$. Sets corresponding to columns are $\{a_1, b_1, d_1, g_1\}$, $\{a_2, b_2, d_2, h_2\}$, $\{a_3, b_3, e_3, i_3\}$, $\{a_4, c_4, e_4, j_4\}$, $\{a_5, c_5, f_5, k_5\}$, $\{a_6, c_6, f_6, l_6\}$.

Optimal cover chooses 6 sets corresponding to 6 columns (say, $k = 6$) but the greedy algorithm chooses sets corresponding to rows (i.e., $\frac{k}{6} + \frac{k}{3} + \frac{k}{2} + \frac{k}{1} < k \log k$).

Any set consisting of arbitrary chosen convex components may not form a fan as every fan consists of contiguous convex components. Therefore, constructing any example where the greedy algorithm takes $O(\log n)$ times optimal does not seem to be possible.



Conjecture: (Ghosh 1986) Approximation algorithms are expected to yield solutions within a constant factor of the optimal.

Lower bound on approximation ratio

Regarding the lower bound on the approximation ratio for the problems of minimum vertex, point and edge guards in simple polygons, it has been shown that these problems are APX-hard using gap-preserving reductions from 5-OCCURRENCE-MAX-3-SAT.

This means that for each of these problems, there exists a constant $\epsilon > 0$ such that an approximation ratio of $1 + \epsilon$ cannot be guaranteed by any polynomial time approximation algorithm unless $P = NP$.

The above statement implies that there may be approximation algorithms for these problems whose approximation ratios are not small constants.

1. S. Eidenbenz C. Stamm and P. Widmayer, *Inapproximability Results for Guarding Polygons and Terrains*, Algorithmica, 31 (2000), 79-113.

On the other hand, for polygons with holes, these problems cannot be approximated by a polynomial time algorithm with ratio $((1 - \epsilon)/12)(ln \quad n)$ for any $(\epsilon \quad > \quad 0)$, unless $NP \subseteq TIME(n^{O(loglogn)})$. The results are obtained by using gap-preserving reductions from the SET COVER problem.

Open problems

Design approximation algorithms for vertex, edge and point guards problems in simple polygons which yield solutions within a constant factor of the optimal.